Set theory and topology

An introduction to the foundations of analysis [1]

Part II: Topology – Fundamental notions

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Abstract

We provide a formal introduction into the classic theorems of general topology and its axiomatic foundations in set theory. In this second part we introduce the fundamental concepts of topological spaces, convergence, and continuity, as well as their applications to real numbers. Various methods to construct topological spaces are presented.

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Remark

This is the second part of a series of articles on the foundations of analysis, cf. [Nagel]. For the Preface and Chapters 1 to 4 see Part I.

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Chapter 5

Topologies and filters

5.1 Set systems

In this Section we introduce three basic functions on set systems, that are used in many places subsequently. Each of these functions is defined with respect to a given set X as a function that maps every subsystem of $\mathcal{P}(X)$ on a—generally larger—subsystem of $\mathcal{P}(X)$. A fourth function is introduced at the end of this Section, which is used in the context of neighborhood system in Section 5.4.
The introduction of the following new symbol turns out to be useful.

Definition 5.1
Given a set X, we write $A \sqsubset X$ if $A \subset X$ and A is finite. ■

Definition 5.2
Given a set X, we define the following functions:

$$\Psi_X : \mathcal{P}^2(X) \longrightarrow \mathcal{P}^2(X), \qquad \Psi_X(\mathcal{A}) = \{\bigcap \mathcal{B} \,:\, \mathcal{B} \sqsubset \mathcal{A},\ \mathcal{B} \neq \emptyset\}\,;$$

$$\Theta_X : \mathcal{P}^2(X) \longrightarrow \mathcal{P}^2(X), \qquad \Theta_X(\mathcal{A}) = \{\bigcup \mathcal{B} \,:\, \mathcal{B} \subset \mathcal{A},\ \mathcal{B} \neq \emptyset\}\,;$$

$$\Phi_X : \mathcal{P}^2(X) \longrightarrow \mathcal{P}^2(X), \qquad \Phi_X(\mathcal{A}) = \{B \subset X \,:\, \exists A \in \mathcal{A}\ \ A \subset B\}$$

When the set X we refer to is evident from the context, we also use the short notations Ψ, Θ, and Φ, respectively. ■

That is, for a system $\mathcal{A}$ of subsets of X, $\Psi(\mathcal{A})$ is the system of all finite intersections of members of $\mathcal{A}$, $\Theta(\mathcal{A})$ is the system of all unions of members of $\mathcal{A}$, and $\Phi(\mathcal{A})$ is the system of all subsets of X that contain some member of $\mathcal{A}$.

Remark 5.3

Given a set X, the following equations hold:

(i) $\Psi(\emptyset) = \emptyset\,, \quad \Theta(\emptyset) = \emptyset\,, \quad \Phi(\emptyset) = \emptyset$

(ii) $\Psi \circ \Psi = \Psi\,, \quad \Theta \circ \Theta = \Theta\,, \quad \Phi \circ \Phi = \Phi$

(iii) $\Psi(\{\emptyset\}) = \{\emptyset\}\,, \quad \Theta(\{\emptyset\}) = \{\emptyset\}\,, \quad \Phi(\{\emptyset\}) = \mathcal{P}(X)$

(iv) $\Psi(\{X\}) = \{X\}\,, \quad \Theta(\{X\}) = \{X\}\,, \quad \Phi(\{X\}) = \{X\}$

(v) $\Psi(\{\emptyset, X\}) = \{\emptyset, X\}\,, \quad \Theta(\{\emptyset, X\}) = \{\emptyset, X\}\,, \quad \Phi(\{\emptyset, X\}) = \mathcal{P}(X)$

■

The identities in Remark 5.3 (ii) say that Ψ, Θ, and Φ are projective.
The composition of two of the functions is not commutative. However, we have the following result.

Lemma 5.4

Given a set X, we have for every $\mathcal{A} \subset \mathcal{P}(X)$:

$$\Psi\,\Theta\,(\mathcal{A}) \subset \Theta\,\Psi\,(\mathcal{A})\,, \qquad \Psi\,\Phi\,(\mathcal{A}) \subset \Phi\,\Psi\,(\mathcal{A})$$

The maps $(\Theta\,\Psi)$ and $(\Phi\,\Psi)$ are projective.

Proof. In order to prove the first claim, let $A = \bigcap_{i=1}^{n} \bigcup \{A_{ij} \,:\, j \in J_i\}$ where $n \in \mathbb{N}$, $n > 0$, and, for every $i \in \mathbb{N}$, $1 \le i \le n$, J_i is an index set and $A_{ij} \in \mathcal{A}$ $(j \in J_i)$. We have

$$A = \bigcup \Big\{ \bigcap\nolimits_{k=1}^{n} A_{kj(k)} \,:\, j \in \bigtimes\nolimits_{i=1}^{n} J_i \Big\} \in \Theta\,\Psi\,(\mathcal{A})$$

To show the second claim, let $n \in \mathbb{N}$, $n > 0$, and for every $i \in \mathbb{N}$, $1 \le i \le n$, let $A_i \in \mathcal{A}$ and B_i be a set with $A_i \subset B_i$. Further let $B = \bigcap_{i=1}^{n} B_i$. It follows that

$B \supset \bigcap_{i=1}^{n} A_i$, and thus $B \in \Phi\,\Psi\,(\mathcal{A})$.
Now the last claim clearly follows. □

Lemma 5.5

Given a set X, $(\mathcal{P}^2(X), \subset)$ is an ordered space in the sense of "$\leq$". In particular, $\subset$ is a reflexive pre-ordering. The maps Ψ, Θ, and Φ as well as their compositions are $\subset$-increasing. For every $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ we have:

(i) $\mathcal{A} \subset_{\Psi} \mathcal{B} \iff \Psi(\mathcal{A}) \subset \Psi(\mathcal{B}) \iff \mathcal{A} \subset \Psi(\mathcal{B})$
$\iff \forall A \in \mathcal{A} \quad \exists \mathcal{G} \sqsubset \mathcal{B} \quad \mathcal{G} \neq \emptyset,\ A = \bigcap \mathcal{G}$

(ii) $\mathcal{A} \subset_{\Theta} \mathcal{B} \iff \Theta(\mathcal{A}) \subset \Theta(\mathcal{B}) \iff \mathcal{A} \subset \Theta(\mathcal{B})$
$\iff \forall A \in \mathcal{A} \quad \exists \mathcal{H} \subset \mathcal{B} \quad \mathcal{H} \neq \emptyset,\ A = \bigcup \mathcal{H}$

(iii) $\mathcal{A} \subset_{\Phi} \mathcal{B} \iff \Phi(\mathcal{A}) \subset \Phi(\mathcal{B}) \iff \mathcal{A} \subset \Phi(\mathcal{B})$
$\iff \forall A \in \mathcal{A} \quad \exists B \in \mathcal{B} \quad A \supset B$

(iv) $\mathcal{A} \subset_{\Theta\Psi} \mathcal{B} \iff \Theta\,\Psi\,(\mathcal{A}) \subset \Theta\,\Psi\,(\mathcal{B}) \iff \mathcal{A} \subset \Theta\,\Psi\,(\mathcal{B})$
$\iff \forall A \in \mathcal{A} \quad \exists \mathcal{H} \subset \Psi(\mathcal{B}) \quad \mathcal{H} \neq \emptyset,\ A = \bigcup \mathcal{H}$

(v) $\mathcal{A} \subset_{\Phi\Psi} \mathcal{B} \iff \Phi\,\Psi\,(\mathcal{A}) \subset \Phi\,\Psi\,(\mathcal{B}) \iff \mathcal{A} \subset \Phi\,\Psi\,(\mathcal{B})$
$\iff \forall A \in \mathcal{A} \quad \exists \mathcal{G} \sqsubset \mathcal{B} \quad \mathcal{G} \neq \emptyset,\ A \supset \bigcap \mathcal{G}$

Proof. In each case the first equivalence is true by definition of the respective relation, cf. Definition 2.88.
To see the second equivalence notice that Ψ, Θ, and Φ are projective by Remark 5.3. The compositions $(\Theta\,\Psi)$ and $(\Phi\,\Psi)$ are projective by Lemma 5.4. The second equivalence follows by Lemma 2.91.

The third equivalence in each case is a consequence of the definition of the maps.

□

Definition 5.6

Given a set X, the function Φ'_X is defined by

$$\Phi'_X : \mathcal{P}\big(X \times \mathcal{P}(X)\big) \longrightarrow \mathcal{P}\big(X \times \mathcal{P}(X)\big) \ ,$$

$$\Phi'_X(R) = \big\{(x,B) \in X \times \mathcal{P}(X) \ : \ \exists A \subset X \ \ (x,A) \in R,\ A \subset B\big\}$$

When X is evident from the context, we also use the short notation Φ' for Φ'_X.

■

Lemma 5.7

Given a set X, $x \in X$, $A \subset X$, and $R \subset X \times \mathcal{P}(X)$, we have:

(i) $(\Phi'(R))\,\{x\} = \Phi(R\,\{x\})$

(ii) $(\Phi'(R))\,[A] = \Phi(R\,[A])$

(iii) $(\Phi'(R))\,\langle A\rangle \supset \Phi(R\langle A\rangle)$

Proof. (i) and (ii) clearly hold. To prove (iii) notice that

$$(\Phi'(R))\,\langle A\rangle \ = \bigcap\nolimits_{x\in A} (\Phi'(R))\,\{x\} \ = \bigcap\nolimits_{x\in A} \Phi(R\,\{x\})$$

$$= \big\{B \subset X \ : \ \forall x \in A \quad \exists B_x \subset B \quad (x,B_x) \in R\big\}$$

$$\supset \big\{B \subset X \ : \ \exists C \subset B \quad \forall x \in A \quad (x,C) \in R\big\} \ = \ \Phi(R\langle A\rangle)$$

□

Lemma 5.8
Given a set X, the pair $(\mathcal{P}(X \times \mathcal{P}(X)), \subset)$ is an ordered space in the sense of "$\leq$". In particular, $\subset$ is a reflexive pre-ordering on $\mathcal{P}(X \times \mathcal{P}(X))$. The map Φ' is $\subset$-increasing and projective. For every $R, S \subset X \times \mathcal{P}(X)$ we have:

$$R \subset_{\Phi'} S \quad \Longleftrightarrow \quad \Phi'(R) \subset \Phi'(S) \quad \Longleftrightarrow \quad R \subset \Phi'(S)$$

$$\Longleftrightarrow \quad \forall (x, A) \in R \quad \exists (y, B) \in S \quad x = y,\, A \supset B$$

$$\Longleftrightarrow \quad \forall x \in X \quad (\Phi'(R))\{x\} \subset (\Phi'(S))\{x\}$$

$$\Longleftrightarrow \quad \forall x \in X \quad \Phi(R\{x\}) \subset \Phi(S\{x\})$$

$$\Longleftrightarrow \quad \forall x \in X \quad R\{x\} \subset_{\Phi} S\{x\}$$

Proof. Exercise. □

5.2 Topologies, bases, subbases

We start with the definition of a topological space.

Definition 5.9
Given a set X, a system $\mathcal{T} \subset \mathcal{P}(X)$ is called **topology on** X if it has all of the following properties:

(i) $\emptyset, X \in \mathcal{T}$

(ii) $\forall \mathcal{G} \subset \mathcal{T} \quad \mathcal{G} \neq \emptyset \implies \bigcup \mathcal{G} \in \mathcal{T}$

(iii) $\forall A, B \in \mathcal{T} \quad A \cap B \in \mathcal{T}$

The pair $\xi = (X, \mathcal{T})$ is called **topological space**. The members of $\mathcal{T}$ are called **ξ-open** or **$\mathcal{T}$-open**. They are also called **open** if the topology is evident from the context. A set $B \subset X$ is called **ξ-closed** if $X \setminus B$ is ξ-open. If the set X is evident from the context, we also say that B is **$\mathcal{T}$-closed**. If the set X and the topology $\mathcal{T}$ are both evident from the context, we also say that B is **closed**. ■

Notice that property (ii) in Definition 5.9 is equivalent to $\Theta(\mathcal{T}) = \mathcal{T}$. By property (iii) it follows that $\bigcap \mathcal{H} \in \mathcal{T}$ for every $\mathcal{H} \sqsubset \mathcal{T}$ with $\mathcal{H} \neq \emptyset$ by the Induction principle. Therefore property (iii) is equivalent to $\Psi(\mathcal{T}) = \mathcal{T}$. Hence the system of topologies on X contains precisely the fixed points of Θ and Ψ that additionally satisfy property (i).
We now define several simple topologies that serve as examples throughout the text.

Lemma and Definition 5.10
Given a set X, each of the following systems of subsets is a topology on X:

(i) $\mathcal{T}_{\text{dis}} = \mathcal{P}(X)$ is called **discrete topology**.

(ii) $\mathcal{T}_{\text{in}} = \{\emptyset, X\}$ is called **indiscrete topology**.

(iii) $\mathcal{T}_{\text{cf}} = \{A \subset X \ : \ A^c \text{ is finite}\} \cup \{\emptyset\}$ is called **cofinite topology**.

(iv) $\mathcal{T}_{\text{cc}} = \{A \subset X \ : \ A^c \text{ is countable}\} \cup \{\emptyset\}$ is called **cocountable topology**.

Proof. Exercise. □

Example 5.11
Let X be a set and $A_n \subset X$ $(n \in \mathbb{N})$ such that $A_n \subset A_{n+1}$ $(n \in \mathbb{N})$ and $\bigcup_{n\in\mathbb{N}} A_n = X$. Then $\mathcal{A} = \{A_n : n \in \mathbb{N}\} \cup \{\emptyset, X\}$ is a topology on X. ■

The analogues of properties (i) to (iii) in Definition 5.9 hold for the system of closed sets as follows.

Lemma 5.12
Let $\xi = (X, \mathcal{T})$ be a topological space. The system $\mathcal{C}$ of all ξ-closed sets has the following properties:

(i) $\emptyset, X \in \mathcal{C}$

(ii) $\forall\, \mathcal{G} \subset \mathcal{C} \quad \mathcal{G} \neq \emptyset \implies \bigcap \mathcal{G} \in \mathcal{C}$

(iii) $\forall A, B \in \mathcal{C} \quad A \cup B \in \mathcal{C}$

Proof. Exercise. □

Clearly, property (iii) in Lemma 5.12 implies that $\bigcup \mathcal{H} \in \mathcal{C}$ for every $\mathcal{H} \sqsubset \mathcal{C}$ with $\mathcal{H} \neq \emptyset$ by the Induction principle (or by the analogue for open sets).

The following Lemma demonstrates that for a given system $\mathcal{C}$ of subsets of X one may first confirm that $\mathcal{C}$ is the system of all closed subsets for some topology on X and then construct the topology from $\mathcal{C}$.

Lemma 5.13
Given a set X and a system $\mathcal{C}$ of subsets of X satisfying (i) to (iii) in Lemma 5.12, the system $\mathcal{T} = \{B^c : B \in \mathcal{C}\}$ is a topology on X, and $\mathcal{C}$ is the system of all $\mathcal{T}$-closed sets.

Proof. Exercise. □

We often encounter more than one topology on the same set. The notions in the following Definition are useful in this case.

Definition 5.14
Let X be a set and $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. If $\mathcal{T}_2 \subset \mathcal{T}_1$, then $\mathcal{T}_1$ is called **finer than** $\mathcal{T}_2$, and $\mathcal{T}_2$ is called **coarser than** $\mathcal{T}_1$. If $\mathcal{T}_1 \subset \mathcal{T}_2$ or $\mathcal{T}_2 \subset \mathcal{T}_1$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are called **comparable**. If $\mathcal{T}_2 \subset \mathcal{T}_1$ and $\mathcal{T}_1 \neq \mathcal{T}_2$, then $\mathcal{T}_1$ is called **strictly finer than** $\mathcal{T}_2$, and $\mathcal{T}_2$ is called **strictly coarser than** $\mathcal{T}_1$. ■

Remark 5.15
Let $(X, \mathcal{T})$ be a topological space. Then we have $\mathcal{T}_{\text{in}} \subset \mathcal{T} \subset \mathcal{T}_{\text{dis}}$. ■

Lemma 5.16
Given an uncountable set X, the topologies defined in Lemma 5.10 (i)–(iv) obey $\mathcal{T}_{\text{in}} \subset \mathcal{T}_{\text{cf}} \subset \mathcal{T}_{\text{cc}} \subset \mathcal{T}_{\text{dis}}$, and no two of them are identical.

Proof. Exercise. □

The following notation allows us a more formal treatment in the sequel.

Definition 5.17

Given a set X, the system of all topologies on X is denoted by $\mathscr{T}(X)$. ■

Lemma and Definition 5.18

Let X be a set, $\mathscr{A} \subset \mathscr{T}(X)$, and $\mathcal{T} \in \mathscr{A}$. The pair $(\mathscr{A}, \subset)$ is a space ordered in the sense of "$\leq$". $\mathcal{T}$ is called **finest** (**coarsest**) topology of $\mathscr{A}$ if it is a maximum (minimum) of $\mathscr{A}$. $\mathscr{A}$ has at most one finest and at most one coarsest topology.

Proof. Exercise. □

Remark 5.19

The discrete topology on X is the finest member of $\mathscr{T}(X)$, and thus it is an upper bound of any subsystem $\mathscr{A} \subset \mathscr{T}(X)$. The indiscrete topology on X is the coarsest member of $\mathscr{T}(X)$, and thus it is a lower bound of any subsystem $\mathscr{A} \subset \mathscr{T}(X)$. ■

It is proven below that for a given set X the supremum and the infimum of every system of topologies $\mathscr{A} \subset \mathscr{T}(X)$ exist and are unique. In regard to Remark 5.19 this is equivalent to the least upper bound property of the ordering $\subset$ on $\mathscr{T}(X)$. The supremum of $\mathscr{A}$, which is the coarsest topology that is finer than every $\mathcal{T} \in \mathscr{A}$, is determined in Corollary 7.4. The infimum of $\mathscr{A}$, which is the finest topology that is coarser than every $\mathcal{T} \in \mathscr{A}$, is determined in Corollary 7.48.
It is often convenient to think of a topology as a system of sets that is, in a sense, "generated" by a subsystem of open sets. The first step in this direction is to determine a subsystem of open sets such that every open set can be written as a union of members of such a generating system. Such a generating system is called a base for the topology. Below we consider a yet smaller subsystem, called subbase.

Definition 5.20
Given a topological space $(X, \mathcal{T})$, a system $\mathcal{B} \subset \mathcal{T}$ is called **base for** $\mathcal{T}$ if $\mathcal{T} = \Theta(\mathcal{B})$, i.e. if the system of all unions of members of $\mathcal{B}$ is identical to $\mathcal{T}$. We also say that $\mathcal{B}$ **generates** $\mathcal{T}$. ■

For a given topology there generally exists more than one base generating it.

Example 5.21
Given a set X, the system $\mathcal{B} = \{\{x\} : x \in X\}$ of all singletons is a base for the discrete topology $\mathcal{T}_{\text{dis}}$ on X. Every other base for $\mathcal{T}_{\text{dis}}$ contains $\mathcal{B}$. ■

Example 5.22
Given a set X, the only base for the indiscrete topology $\mathcal{T}_{\text{in}}$ on X is $\mathcal{T}_{\text{in}}$ itself. ■

Lemma 5.23
Given a topological space $(X, \mathcal{T})$, a system $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ iff for every $U \in \mathcal{T}$ and $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Proof. Exercise. □

The following Lemma provides a characterization of a system of subsets of X to be a base for some topology.

Lemma and Definition 5.24

Let X be a set and $\emptyset \neq \mathcal{B} \subset \mathcal{P}(X)$. $\mathcal{B}$ is a base for some topology on X iff it satisfies all of the following conditions:

(i) $\emptyset \in \mathcal{B}$

(ii) $X = \bigcup \mathcal{B}$

(iii) $\forall A, B \in \mathcal{B} \quad \exists \mathcal{C} \subset \mathcal{B} \quad \mathcal{C} \neq \emptyset \ \wedge \ A \cap B = \bigcup \mathcal{C}$

In this case, $\mathcal{B}$ is also called a **topological base on** X. The topology generated by $\mathcal{B}$ is unique.

Proof. Assume that (i) to (iii) hold, and let $\mathcal{T} = \Theta(\mathcal{B})$. It follows by Definition 5.9 and Lemma 1.34 that $\mathcal{T}$ is a topology on X. Clearly $\mathcal{B}$ generates $\mathcal{T}$. The converse implication and the uniqueness of the topology generated by $\mathcal{B}$ are obvious. □

Notice that we have excluded the case $\mathcal{B} = \emptyset$ in Lemma and Definition 5.24 only because property (ii) has to be well-defined. If a system $\mathcal{B}$ satisfies property (i), this obviously implies $\mathcal{B} \neq \emptyset$.

We may compare two topological bases on the same set X with each other similarly as we compare two topologies. We even refer in our definitions to the corresponding notions for the comparison of two topologies.

Definition 5.25

Let $\mathcal{A}$ and $\mathcal{B}$ be two topological bases on a set X. $\mathcal{B}$ is called **finer**, **coarser**, **strictly finer**, **strictly coarser than** $\mathcal{B}$ if the generated topologies $\Theta(\mathcal{A})$ and $\Theta(\mathcal{B})$ have the respective property. $\mathcal{A}$ and $\mathcal{B}$ are called **comparable** if $\Theta(\mathcal{A})$ and $\Theta(\mathcal{B})$ are comparable. ■

Notice that every topology on a set X is a base for itself. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set X, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ in the sense of Definition 5.14 iff $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ in the sense of Definition 5.25, etc. That is Definitions 5.14 and 5.25 are consistent when referring to two topologies in both cases.

Definition 5.26

Given a set X, the system of all topological bases on X is denoted by $\mathscr{T}_B(X)$. ■

Remark 5.27

Given a set X, we have $\mathscr{T}(X) \subset \mathscr{B}(X)$ and $\mathscr{B}(X) = \Theta^{-1}\left[\mathscr{T}(X)\right]$. For every $\mathcal{T} \in \mathscr{T}(X)$, the system of all bases for $\mathcal{T}$ is given by $\Theta^{-1}\left\{\mathcal{T}\right\}$. ■

When we compare two topologies on X, we use the ordering $\subset$ on $\mathscr{T}(X)$. The direct comparison of topological bases requires another pre-ordering on $\mathcal{P}^2(X)$ whose properties are analysed in Lemma 5.5.

Remark 5.28

Given a set X, the pair $\left(\mathscr{B}(X), \subset_\Theta\right)$ is a pre-ordered space with a reflexive relation. Let $\mathcal{A}, \mathcal{B} \in \mathscr{B}(X)$. The following statements are true:

(i) $\mathcal{A}$ is finer than $\mathcal{B} \iff \mathcal{B} \subset_\Theta \mathcal{A}$

(ii) $\mathcal{A}$ is strictly finer than $\mathcal{B} \iff \left(\mathcal{B} \subset_\Theta \mathcal{A}\right) \wedge \neg\left(\mathcal{A} \subset_\Theta \mathcal{B}\right)$

(iii) $\mathcal{A}$ and $\mathcal{B}$ are comparable $\iff \left(\mathcal{B} \subset_\Theta \mathcal{A}\right) \vee \left(\mathcal{A} \subset_\Theta \mathcal{B}\right)$

■

Remark 5.29

Let X be a set and $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$. By Lemma 5.5 (ii) we have

$$\Theta(\mathcal{A}) = \Theta(\mathcal{B}) \quad \Longleftrightarrow \quad \big(\mathcal{B} \subset_\Theta \mathcal{A}\big) \wedge \big(\mathcal{A} \subset_\Theta \mathcal{B}\big)$$

If one side is true (and hence both sides are true) and $\mathcal{A}$ is a topological base, then also $\mathcal{B}$ is a topological base.

Notice that $\Theta(\mathcal{A}) = \Theta(\mathcal{B})$ need not imply $\mathcal{A} = \mathcal{B}$. ■

The following is a counterexample.

Example 5.30

Let X be an infinite set. We may define systems

$$\mathcal{B}_n = \{B_{nk} \subset X \,:\, k \in \mathbb{N},\, 1 \leq k \leq 2^n\} \quad (n \in \mathbb{N})$$

with the following properties:

(i) $B_{nk} \cap B_{nl} = \emptyset \quad (n, k, l \in \mathbb{N};\;\; 1 \leq k, l \leq 2^n;\;\; k \neq l)$

(ii) $\bigcup \mathcal{B}_n = X \quad (n \in \mathbb{N})$

(iii) $B_{nk} = B_{(n+1)(2k-1)} \cup B_{(n+1)(2k)} \quad (n, k \in \mathbb{N};\;\; 1 \leq k \leq 2^n)$

Then each of the systems

$$\mathcal{C} = \bigcup \{\mathcal{B}_n \,:\, n \text{ is odd}\} \cup \{\emptyset\}\,, \quad \mathcal{D} = \bigcup \{\mathcal{B}_n \,:\, n \text{ is even}\} \cup \{\emptyset\}$$

is a topological base on X, and $\Theta(\mathcal{C}) = \Theta(\mathcal{D})$. However, we have $\mathcal{C} \neq \mathcal{D}$. ■

Definition 5.31

We say that a topological space $(X, \mathcal{T})$ or the topology $\mathcal{T}$ is **second countable** if there exists a countable base for $\mathcal{T}$. ■

We define below what we mean by "first countable" topological space or topology. This definition is based on the notion of neighborhood to be introduced in Section 5.4.

Example 5.32
Given an uncountable set X, the discrete topology on X is not second countable.
■

In the same way as for the system of all open sets, there is a possibility to "generate" the system of all closed sets from an appropriate subsystem that we call "base for the closed sets". It is a system such that every closed set is an intersection of its members.

Definition 5.33
Let $\xi = (X, \mathcal{T})$ be a topological space and $\mathcal{C}$ the system of all ξ-closed sets. A system $\mathcal{D} \subset \mathcal{C}$ is called **base for** $\mathcal{C}$ or **base for the** ξ**-closed sets** if $\mathcal{C} = \{\bigcap \mathcal{G} : \mathcal{G} \subset \mathcal{D},\ \mathcal{G} \neq \emptyset\}$. When the set X is evident from the context, $\mathcal{C}$ is also called **base for the** $\mathcal{T}$**-closed sets**. When the set X as well as the topology $\mathcal{T}$ are evident from the context, $\mathcal{C}$ is also called **base for the closed sets**. ■

The base for a topology and the base for the closed sets are related by complementation as follows.

Lemma 5.34
Let $(X, \mathcal{T})$ be a topological space, $\mathcal{C}$ the system of all closed sets, and $\mathcal{B} \subset \mathcal{P}(X)$. $\mathcal{B}$ is a base for $\mathcal{T}$ iff $\{B^c : B \in \mathcal{B}\}$ is a base for $\mathcal{C}$.

Proof. Exercise. □

The analogue of Lemma 5.24 for the system of all closed sets is stated in the following Lemma.

Lemma 5.35

Let X be a set and $\emptyset \neq \mathcal{D} \subset \mathcal{P}(X)$. $\mathcal{D}$ is a base for the $\mathcal{T}$-closed sets where $\mathcal{T}$ is some topology on X iff it satisfies all of the following conditions:

(i) $X \in \mathcal{D}$

(ii) $\emptyset = \bigcap \mathcal{D}$

(iii) $\forall A, B \in \mathcal{D} \quad \exists \mathcal{E} \subset \mathcal{D} \quad \mathcal{E} \neq \emptyset \;\wedge\; A \cup B = \bigcap \mathcal{E}$

Proof. This follows by Lemmas 5.24 and 5.34. □

Again, notice that the case $\mathcal{D} = \emptyset$ is excluded in Lemma 5.35 in order for the intersection in property (ii) to be well-defined. If a system $\mathcal{D}$ satisfies property (i), this obviously implies $\mathcal{D} \neq \emptyset$.
As announced before we are often able to "generate" a topological base from an appropriate smaller subsystem, called a "subbase".

Definition 5.36

Given a topological space $(X, \mathcal{T})$, a system $\mathcal{S} \subset \mathcal{T}$ is called **subbase for** $\mathcal{T}$ if the system $\Psi(\mathcal{S})$ is a base for $\mathcal{T}$. We also say that $\mathcal{S}$ **generates** $\mathcal{T}$. ■

Generally, for a given topological base, there exist more than one subbase generating it.
There is a simple criterion to probe whether a given system of subsets of X is a subbase for some topology on X as shown in the following Lemma. To this end the notion of "finite intersection property" is introduced.

Definition 5.37

Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$. We say that $\mathcal{C}$ **has the finite intersection property** if $\emptyset \notin \Psi(\mathcal{C})$, i.e. for every $\mathcal{H} \sqsubset \mathcal{C}$ with $\mathcal{H} \neq \emptyset$ we have $\bigcap \mathcal{H} \neq \emptyset$. ■

Lemma 5.38

Let X be a set and $\emptyset \neq \mathcal{S} \subset \mathcal{P}(X)$. $\mathcal{S}$ is a subbase for some topology $\mathcal{T}$ on X iff both of the following statements are true:

(i) $\mathcal{S}$ does not have the finite intersection property.

(ii) $X = \bigcup \mathcal{S}$

In this case, $\mathcal{S}$ is also called a **topological subbase on** X, and the topology generated by $\mathcal{S}$ is unique. Furthermore, $\mathcal{T}$ is the coarsest topology on X that contains $\mathcal{S}$.

Proof. The first claim is easy to verify (exercise).
Now let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X and $\mathcal{S}$ a subbase both for $\mathcal{T}_1$ and for $\mathcal{T}_2$. By Definition 5.36 $\Psi(\mathcal{S})$ is a base both for $\mathcal{T}_1$ and for $\mathcal{T}_2$. Hence, $\mathcal{T}_1 = \mathcal{T}_2$ by Lemma 5.24.
To prove the last claim, let $\mathcal{S}$ be a topological subbase. Further let $\mathcal{T}_1$ be a topology on X with $\mathcal{S} \subset \mathcal{T}_1$. It follows that $\Theta\,\Psi\,(\mathcal{S}) \subset \mathcal{T}_1$. □

Notice again that we have excluded the case $\mathcal{S} = \emptyset$ in order for property (ii) to be well-defined. Of course, if a system $\mathcal{S}$ satisfies property (i), this implies $\mathcal{S} \neq \emptyset$.
Notice that for any system $\mathcal{S} \subset \mathcal{P}(X)$ the system $\mathcal{S} \cup \{\emptyset, X\}$ is a topological subbase. Alternatively, one could use the conventions that an intersection of an empty system of sets is identical to X and that a union of an empty system of sets is identical to $\emptyset$. In such a framework any arbitrary system of subsets of X would be a subbase for some topology on X. In this account the intersection or

union of an empty system of sets is not defined.

We now state a criterion for the second countability of a topology.

Lemma 5.39

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{S}$ a topological subbase for $\mathcal{T}$. If $\mathcal{S}$ is countable, then $\mathcal{T}$ is second countable.

Proof. This follows by Lemma 3.71. □

Next we introduce some notions for the comparison of two subbases on the same set.

Definition 5.40

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two topological subbases on a set X. $\mathcal{S}_1$ is called **finer**, **coarser**, **strictly finer**, **strictly coarser than** $\mathcal{S}_2$ if the topological bases $\Psi(\mathcal{S}_1)$ and $\Psi(\mathcal{S}_2)$ have the respective property. $\mathcal{S}_1$ and $\mathcal{S}_2$ are called **comparable** if $\Psi(\mathcal{S}_1)$ and $\Psi(\mathcal{S}_2)$ are comparable. ■

Note that the notions defined in Definition 5.40 are well-defined since every topological subbase generates a unique topological base. Furthermore, every topological base is a subbase itself. When comparing two bases, no confusion can arise whether this comparison is done in the sense of Definition 5.25 or Definition 5.40 since Definition 5.40 refers to the corresponding notions for bases.

Definition 5.41

Given a set X, the system of all topological subbases on X is denoted by $\mathscr{S}(X)$. ■

Remark 5.42

Given a set X, we have $\mathscr{B}(X) \subset \mathscr{S}(X)$ and $\mathscr{S}(X) = \Psi^{-1}[\mathscr{B}(X)]$. For every $\mathcal{T} \in \mathscr{T}(X)$, the system of all subbases for $\mathcal{T}$ is given by $(\Theta\,\Psi)^{-1}\{\mathcal{T}\}$. ∎

Similarly to the case of topological bases in Remark 5.28, the comparison between two topological subbases may be expressed in terms of a pre-ordering as stated by the following remark.

Remark 5.43

Given a set X, the pair $\big(\mathscr{S}(X), \subset_{\Theta\Psi}\big)$ is a pre-ordered space with a reflexive relation. Let $\mathcal{A}, \mathcal{B} \in \mathscr{S}(X)$. The following statements are true by Lemma 5.5 (iv):

(i) $\mathcal{A}$ is finer than $\mathcal{B} \iff \mathcal{B} \subset_{\Theta\Psi} \mathcal{A}$

(ii) $\mathcal{A}$ is strictly finer than $\mathcal{B} \iff \big(\mathcal{B} \subset_{\Theta\Psi} \mathcal{A}\big) \wedge \neg\big(\mathcal{A} \subset_{\Theta\Psi} \mathcal{B}\big)$

(iii) $\mathcal{A}$ and $\mathcal{B}$ are comparable $\iff \big(\mathcal{B} \subset_{\Theta\Psi} \mathcal{A}\big) \vee \big(\mathcal{A} \subset_{\Theta\Psi} \mathcal{B}\big)$

∎

Remark 5.44

Let X be a set and $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$. By Lemma 5.5 (iv) we have

$$\Theta\,\Psi(\mathcal{A}) = \Theta\,\Psi(\mathcal{B}) \iff \big(\mathcal{B} \subset_{\Theta\Psi} \mathcal{A}\big) \wedge \big(\mathcal{A} \subset_{\Theta\Psi} \mathcal{B}\big)$$

In this case, if $\mathcal{A}$ is a topological subbase, then also $\mathcal{B}$ is a topological subbase and the topologies generated by $\mathcal{A}$ and $\mathcal{B}$ are the same. Furthermore, notice that $\Theta\,\Psi(\mathcal{A}) = \Theta\,\Psi(\mathcal{B})$ need not imply $\mathcal{A} = \mathcal{B}$. ∎

Definition 5.45
Let $\xi = (X, \mathcal{T})$ be a topological space and $\mathcal{C}$ the system of all ξ-closed sets. A system $\mathcal{S} \subset \mathcal{C}$ is called **subbase for $\mathcal{C}$** or **subbase for the ξ-closed sets** if $\{\bigcup \mathcal{H} : \mathcal{H} \sqsubset \mathcal{S},\ \mathcal{H} \neq \emptyset\}$ is a base for $\mathcal{C}$. If the set X is evident from the context, $\mathcal{C}$ is also called **subbase for the $\mathcal{T}$-closed sets**. If the set X and the topology $\mathcal{T}$ are evident, $\mathcal{C}$ is called **subbase for the closed sets**. ■

Lemma 5.46
Let $\xi = (X, \mathcal{T})$ be a topological space, $\mathcal{C}$ the system of all ξ-closed sets, and $\mathcal{S} \subset \mathcal{P}(X)$. Then $\mathcal{S}$ is a subbase for $\mathcal{T}$ iff the system $\{S^c : S \in \mathcal{S}\}$ is a subbase for $\mathcal{C}$.

Proof. Exercise. □

Lemma 5.47
Let X be a set and $\emptyset \neq \mathcal{S} \subset \mathcal{P}(X)$. $\mathcal{S}$ is subbase for the $\mathcal{T}$-closed sets, where $\mathcal{T}$ is some topology on X, iff both of the following conditions are satisfied:

(i) There exists $\mathcal{H} \sqsubset \mathcal{S}$ such that $\mathcal{H} \neq \emptyset$ and $X = \bigcup \mathcal{H}$.

(ii) $\emptyset = \bigcap \mathcal{S}$

Proof. This follows from Lemmas 5.38 and 5.46. □

5.3 Filters

So far, in this chapter, we have considered systems of sets whose members are subsets of a common set X as well as properties between members of such systems. Thereby we have not referred to any specific points of X. In much of the following we consider particular points, values of maps at specific points, systems of sets that contain specific points etc. In particular this is the case in all circumstances where convergence is examined, a notion that is introduced in Chapter 6. To this end, in this section the notion of filter is introduced as a fundamental concept. Note that the definitions and claims in this section do not involve any topology on X.

Definition 5.48

Given a set X, a system $\mathcal{F} \subset \mathcal{P}(X)$ is called a **filter on** X if it has all of the following properties:

(i) $\emptyset \notin \mathcal{F}$

(ii) $X \in \mathcal{F}$

(iii) $\forall A, B \subset X \quad A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$

(iv) $\forall A, B \subset X \quad (A \in \mathcal{F}) \wedge (A \subset B) \implies B \in \mathcal{F}$

■

Properties (i) and (iii) in Definition 5.48 imply that $\mathcal{F}$ has the finite intersection property.

Remark 5.49

Let X be a set and $\mathcal{F} \subset \mathcal{P}(X)$. $\mathcal{F}$ is a filter on X iff all of the following conditions are satisfied:

(i) $\mathcal{F} \neq \emptyset$

(ii) $\mathcal{F} \neq \mathcal{P}(X)$

(iii) $\Psi(\mathcal{F}) = \mathcal{F}$

(iv) $\Phi(\mathcal{F}) = \mathcal{F}$

That is, the filters on X are precisely those systems that are fixed points of both Ψ and Φ and additionally are non-trivial in the sense of (i) and (ii). ■

When comparing two filters on the same set we use the same notions as for topologies.

Definition 5.50

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filters on a set X. If $\mathcal{F}_2 \subset \mathcal{F}_1$, then $\mathcal{F}_1$ is called **finer than** $\mathcal{F}_2$ and $\mathcal{F}_2$ is called **coarser than** $\mathcal{F}_1$. If $\mathcal{F}_2 \subset \mathcal{F}_1$ and $\mathcal{F}_1 \neq \mathcal{F}_2$, then $\mathcal{F}_1$ is called **strictly finer than** $\mathcal{F}_2$ and $\mathcal{F}_2$ is called **strictly coarser than** $\mathcal{F}_1$. If $\mathcal{F}_1 \subset \mathcal{F}_2$ or $\mathcal{F}_2 \subset \mathcal{F}_1$, the filters are called **comparable**. ■

Definition 5.51

Given a set X, the system of all filters on X is denoted by $\mathscr{F}(X)$. ■

Lemma and Definition 5.52

Let X be a set, $\mathscr{A} \subset \mathscr{F}(X)$, and $\mathcal{F} \in \mathscr{A}$. The pair $(\mathscr{A}, \subset)$ is an ordered space in the sense "$\leq$". $\mathcal{F}$ is called **finest** (**coarsest**) filter of $\mathscr{A}$ if it is a maximum (minimum) of $\mathscr{A}$. $\mathscr{A}$ has at most one finest and at most one coarsest filter.

Proof. Exercise. □

Definition 5.53
Given a filter $\mathcal{F}$ on a set X, $\mathcal{F}$ is called **ultrafilter** if it is a weak maximum of $\mathscr{F}(X)$ with respect to the ordering $\subset$, i.e. if there is no filter on X that is strictly finer than $\mathcal{F}$. ■

Given a specific filter it is often convenient to consider only a particular subsystem, called filter base, instead of the entire filter. A filter base is defined by the requirement that every member of the filter contains a member of the filter base. More formally we have the following definition.

Definition 5.54
Given a filter $\mathcal{F}$ on a set X, a subsystem $\mathcal{B} \subset \mathcal{F}$ is called **filter base for** $\mathcal{F}$ if $\mathcal{F} = \Phi(\mathcal{B})$. We also say that $\mathcal{B}$ **generates** $\mathcal{F}$. ■

Notice that this definition is similar to the definition of a base for a topology, see Definition 5.20. The two are related by the concept of neighborhood system, which is described below in Section 5.4.

Lemma and Definition 5.55
Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$. $\mathcal{B}$ is a filter base for some filter on X iff all of the following conditions are satisfied:

(i) $\emptyset \notin \mathcal{B}$

(ii) $\mathcal{B} \neq \emptyset$

(iii) $\forall A, B \in \mathcal{B} \quad \exists C \in \mathcal{B} \quad C \subset A \cap B$

In this case, $\mathcal{B}$ is also called **filter base on** X. The filter generated by $\mathcal{B}$ is unique.

Proof. Assume that (i) to (iii) hold. Then clearly $\mathcal{F} = \Phi(\mathcal{B})$ is a filter on X. The converse implication and the uniqueness of the generated filter are obvious. □

Notice that property (iii) in Lemma and Definition 5.55 is equivalent to $\Psi(\mathcal{B}) \subset \Phi(\mathcal{B})$.
We now define appropriate notions for the comparison of two filter bases.

Definition 5.56
Let X be a set, and $\mathcal{A}$ and $\mathcal{B}$ two filter bases on X. $\mathcal{A}$ is called **finer**, **coarser**, **strictly finer**, **strictly coarser than** $\mathcal{B}$ if $\Phi(\mathcal{A})$ and $\Phi(\mathcal{B})$ have the respective property. $\mathcal{A}$ and $\mathcal{B}$ are called **comparable** if $\Phi(\mathcal{A})$ and $\Phi(\mathcal{B})$ are comparable. ■

This definition relies on the fact that the filter generated by a filter base is unique. Also the fact that every filter is a filter base for itself is taken into account by referring to the properties of filters in the definition.

Definition 5.57
Given a set X, the system of all filter bases on X is denoted by $\mathscr{F}_B(X)$. ■

Remark 5.58
Given a set X, we have $\mathscr{F}(X) \subset \mathscr{F}_B(X)$ and $\mathscr{F}_B(X) = \Phi^{-1}[\mathscr{F}(X)]$. For every $\mathcal{F} \in \mathscr{F}(X)$, the system of all filter bases for $\mathcal{F}$ is given by $\Phi^{-1}\{\mathcal{F}\}$. ■

The comparison of two filter bases on a set X may be expressed by the pre-ordering $\subset_\Phi$.

Remark 5.59
Given a set X, the pair $\left(\mathscr{F}_B(X), \subset_\Phi\right)$ is a pre-ordered space with a reflexive relation. For every $\mathcal{A}, \mathcal{B} \in \mathscr{F}_B(X)$, the following statements are true by Lemma 5.5 (iii)

(i) $\mathcal{A}$ is finer than $\mathcal{B} \iff \mathcal{B} \subset_\Phi \mathcal{A}$

(ii) $\mathcal{A}$ is strictly finer than $\mathcal{B} \iff \left(\mathcal{B} \subset_\Phi \mathcal{A}\right) \wedge \neg\left(\mathcal{A} \subset_\Phi \mathcal{B}\right)$

(iii) $\mathcal{A}$ and $\mathcal{B}$ are comparable $\iff \left(\mathcal{B} \subset_\Phi \mathcal{A}\right) \vee \left(\mathcal{A} \subset_\Phi \mathcal{B}\right)$

■

Remark 5.60
Let X be a set and $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$. By Lemma 5.5 (iii) we have $\Phi(\mathcal{A}) = \Phi(\mathcal{B})$ iff $\mathcal{A} \subset_\Phi \mathcal{B}$ and $\mathcal{B} \subset_\Phi \mathcal{A}$. In this case, if $\mathcal{A}$ is a filter base on X, then $\mathcal{B}$ is also a filter base on X and the two filters generated by $\mathcal{A}$ and $\mathcal{B}$ are the same. Notice however that $\Phi(\mathcal{A}) = \Phi(\mathcal{B})$ need not imply $\mathcal{A} = \mathcal{B}$. ■

Definition 5.61

Given a filter $\mathcal{F}$ on a set X, a point $x \in X$ is called a **cluster point of** $\mathcal{F}$ if $x \in \bigcap \mathcal{F}$. $\mathcal{F}$ is called **free** if it has no cluster points, otherwise it is called **fixed**. Similarly, for a filter base $\mathcal{B}$ on X, a point $x \in X$ is called a **cluster point of** $\mathcal{B}$ if it is a cluster point of $\Phi(\mathcal{B})$. $\mathcal{B}$ is called **free** if it has no cluster points, otherwise it is called **fixed**. ■

Remark 5.62

Given a filter base $\mathcal{B}$ on a set X and a point $x \in X$, x is a cluster point of $\mathcal{B}$ iff $x \in \bigcap \mathcal{B}$. ■

Example 5.63

Given a set X, the system $\mathcal{B} = \{\,]x, \infty[\, : x \in \mathbb{R}\}$ is a free filter base on $\mathbb{R}$. ■

Example 5.64

For every $r \in \,]0, \infty[\,$, let $B_r = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r\right\}$. The system $\mathcal{B} = \{B_r : r \in \,]0, \infty[\,\}$ is a fixed filter base on $\mathbb{R}^2$. ■

Example 5.65

Given a set X, the system $\mathcal{F} = \{X\}$ is a filter. It is called the **indiscrete filter on** X. ■

Lemma 5.66

Given a set X and $A \subset X$ with $A \neq \emptyset$, the system $\mathcal{F} = \{F \subset X : A \subset F\}$ is a fixed filter on X. The system $\{A\}$ is a filter base for $\mathcal{F}$. $\mathcal{F}$ is an ultrafilter iff $A = \{x\}$ for some $x \in X$. In this case it is called the **discrete filter at** x.

Proof. Exercise. □

Lemma and Definition 5.67
Let $\mathcal{A}$ and $\mathcal{B}$ be two filter bases on a set X, where $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$, $B \in \mathcal{B}$. The system $\mathcal{C} = \{A \cap B : A \in \mathcal{A},\, B \in \mathcal{B}\}$ is a filter base. It is called the **filter-base intersection of $\mathcal{A}$ and $\mathcal{B}$**. $\mathcal{C}$ is a supremum of $\{\mathcal{A}, \mathcal{B}\}$ in the pre-ordered space $\big(\mathscr{F}_B(X), \subset_\Phi\big)$. Equivalently, the filter $\Phi(\mathcal{C})$ is the supremum of $\{\Phi(\mathcal{A}), \Phi(\mathcal{B})\}$ in the ordered space $(\mathscr{F}(X), \subset)$, i.e. it is the coarsest filter on X that is finer than $\Phi(\mathcal{A})$ and $\Phi(\mathcal{B})$.

Proof. Let $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$. We may choose $A_3 \in \mathcal{A}$ such that $A_3 \subset A_1 \cap A_2$, and $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$. It follows that

$$A_3 \cap B_3 \;\subset\; A_1 \cap B_1 \cap A_2 \cap B_2$$

Thus $\mathcal{C}$ is a filter base. We clearly have $\mathcal{A} \subset_\Phi \mathcal{C}$ and $\mathcal{B} \subset_\Phi \mathcal{C}$. Assume that $\mathcal{D}$ is a filter base on X with $\mathcal{A} \subset_\Phi \mathcal{D}$ and $\mathcal{B} \subset_\Phi \mathcal{D}$. We show that $\mathcal{C} \subset_\Phi \mathcal{D}$. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We may choose $A', B' \in \mathcal{D}$ such that $A' \subset A$ and $B' \subset B$. There exists $D \in \mathcal{D}$ such that $D \subset A' \cap B'$. Thus we have $D \subset A \cap B$. This shows that $\mathcal{C}$ is a supremum. It clearly follows that $\Phi(\mathcal{C})$ is a supremum, which is unique since $\subset$ is an ordering. ☐

Lemma 5.68
Let X be a set and $\mathcal{B}_i \in \mathscr{F}_B(X)$ $(i \in I)$ where I an index set. Furthermore we define the system

$$\mathcal{B} = \Big\{ \bigcap_{j \in J} B_j \;:\; J \sqsubset I,\; J \neq \emptyset,\; B_j \in \mathcal{B}_j \;(j \in J) \Big\}$$

If $\emptyset \notin \mathcal{B}$, then $\mathcal{B} \in \mathscr{F}_B(X)$. In this case, $\mathcal{B}$ is a supremum of $\{\mathcal{B}_i \;:\; i \in I\}$ in the pre-ordered space $\big(\mathscr{F}_B(X), \subset_\Phi\big)$. Equivalently, the filter $\Phi(\mathcal{B})$ is the supremum of $\{\Phi(\mathcal{B}_i) \;:\; i \in I\}$ in the ordered space $(\mathscr{F}(X), \subset)$, i.e. it is the coarsest filter on X that is finer than $\Phi(\mathcal{B}_i)$ for every $i \in I$.

Proof. Assume that $\emptyset \notin \mathcal{B}$. It follows by Lemma 5.55 that $\mathcal{B}$ is a filter base on X. Clearly, $\mathcal{B}$ is finer than $\mathcal{B}_i$ for every $i \in I$. It remains to show that every filter base on X that is finer than $\mathcal{B}_i$ for every $i \in I$ is also finer than $\mathcal{B}$. Assume that $\mathcal{A}$ is a filter base on X that is finer than $\mathcal{B}_i$ for every $i \in I$, and let $B \in \mathcal{B}$. We may choose $J \sqsubset I$ with $J \neq \emptyset$ and $B_j \in \mathcal{B}_j$ $(j \in J)$ such that $B = \bigcap_{j \in J} B_j$. For every $j \in J$ we may choose $A_j \in \mathcal{A}$ such that $A_j \subset B_j$ by assumption. There exists $A \in \mathcal{A}$ such that $A \subset \bigcap_{j \in J} A_j$. Hence $A \subset B$. This shows that $\mathcal{B}$ is a supremum of $\{\mathcal{B}_i : i \in I\}$. It clearly follows that $\Phi(\mathcal{B})$ is a supremum of $\{\Phi(\mathcal{B}_i) : i \in I\}$, which is unique since $\subset$ is an ordering. □

Notice that the filter base $\mathcal{B}$ defined in Lemma 5.68 is not a direct generalization of Lemma and Definition 5.67, as the index set J may be a singleton. However, it is easy to see that the generated filters are the same if $J \sim 2$.

Definition 5.69
Let $\mathcal{B}$ be a filter base on a set X. $\mathcal{B}$ is called **ultrafilter base** if $\Phi(\mathcal{B})$ is an ultrafilter. ■

The following are a few characterizations of an ultrafilter base.

Lemma 5.70
Let $\mathcal{B}$ be a filter base on a set X. The following statements are equivalent.

(i) $\mathcal{B}$ is an ultrafilter base.

(ii) $\mathcal{B}$ is a weak maximum of $\mathscr{F}_B(X)$ with respect to the pre-ordering $\subset_\Phi$, i.e. there is no filter base on X that is strictly finer than $\mathcal{B}$.

(iii) $\forall A \subset X \quad A \in \Phi(\mathcal{B}) \ \vee \ A^c \in \Phi(\mathcal{B})$

(iv) $\forall A \subset X \quad \{A\} \subset_\Phi \mathcal{B} \ \vee \ \{A^c\} \subset_\Phi \mathcal{B}$

Proof. (i) and (ii) are clearly equivalent.
The equivalence of (iii) and (iv) is obvious as well.
To show that (i) implies (iv), let $A \subset X$ and assume that neither $\{A\} \subset_\Phi \mathcal{B}$ nor $\{A^c\} \subset_\Phi \mathcal{B}$ holds. It follows by Lemma and Definition 5.67 that each of the systems

$$\mathcal{A} = \{B \cap A \,:\, B \in \mathcal{B}\}\,, \qquad \mathcal{C} = \{B \cap A^c \,:\, B \in \mathcal{B}\}$$

is a filter base and that $\mathcal{A}$ and $\mathcal{C}$ are both finer than $\mathcal{B}$. We clearly have $\Phi(\mathcal{A}) \neq \Phi(\mathcal{C})$. Hence at least one of the filter bases is strictly finer than $\mathcal{B}$, which is a contradiction to the fact that $\mathcal{B}$ is an ultrafilter base.
To show that (iv) implies (i), let $\mathcal{A}$ be a filter base on X that is finer than $\mathcal{B}$, and $A \in \mathcal{A}$. Since $\{A^c\} \subset_\Phi \mathcal{B}$ clearly does not hold, we have $\{A\} \subset_\Phi \mathcal{B}$. Therefore $\mathcal{B}$ is finer than $\mathcal{A}$, and hence $\mathcal{B}$ is an ultrafilter base. □

Note that Lemma 5.70 says that ultrafilters are precisely those filters that contain either A or A^c for every $A \subset X$.

Lemma 5.71
Let X be a set. $\mathcal{F}$ is a fixed ultrafilter on X iff there is a point $x \in X$ such that $\mathcal{F} = \{F \subset X \,:\, x \in F\}$.

Proof. If $\mathcal{F}$ is a fixed ultrafilter, it clearly has a unique cluster point by Lemma 5.70 (iii), say x. Let $F \subset X$ with $x \in F$. Since $F^c \notin \mathcal{F}$, we have $F \in \mathcal{F}$ by Lemma 5.70 (iii). This shows that $\mathcal{F} = \{F \subset X \,:\, x \in F\}$.
The converse is obvious. □

Theorem 5.72
For every filter base $\mathcal{B}$ on a set X there exists an ultrafilter base that is finer than $\mathcal{B}$.

Proof. Let $M = \{\mathcal{B}_i : i \in I\} \subset \mathscr{F}_B(X)$ be the set of all filter bases on X that are finer than $\mathcal{B}$ where I is an index set. The relation $\subset_\Phi$ is a pre-ordering on M. Let $L = \{\mathcal{B}_j : j \in J\}$ ($J \subset I$) be a chain and

$$\mathcal{A} = \left\{ \bigcap_{k \in K} B_k \,:\, K \sqsubset J,\ K \neq \emptyset,\ B_k \in \mathcal{B}_k\ (k \in K) \right\}$$

Since $\emptyset \notin \mathcal{A}$, $\mathcal{A}$ is a filter base that is finer than $\mathcal{B}_j$ for every $j \in J$ by Lemma 5.68. Thus $\mathcal{A}$ is an upper bound of L. Let $\mathcal{C}$ be a weak maximum of M according to Theorem 3.56. To prove that $\mathcal{C}$ is an ultrafilter base, assume that $\mathcal{D}$ is a filter base on X with $\mathcal{C} \subset_\Phi \mathcal{D}$. Since $\mathcal{B} \subset_\Phi \mathcal{C}$, we have $\mathcal{B} \subset_\Phi \mathcal{D}$, and thus $\mathcal{D} \in M$. Since $\mathcal{C}$ is a weak maximum of M, $\mathcal{C} \subset_\Phi \mathcal{D}$ implies $\mathcal{D} \subset_\Phi \mathcal{C}$. □

Corollary 5.73
For every filter $\mathcal{F}$ on a set X, there is an ultrafilter $\mathcal{G}$ that is finer than $\mathcal{F}$.

Proof. By Theorem 5.72 we may choose an ultrafilter base $\mathcal{B}$ such that $\mathcal{B}$ is finer than $\mathcal{F}$. Then $\Phi(\mathcal{B})$ is an ultrafilter that is finer than $\mathcal{F}$. □

5.4 Neighborhoods

We briefly outline the major notions defined in this Section. Given a topological space $(X, \mathcal{T})$, a neighborhood U of a point x is defined as a—not necessarily open—set for which there exists an open set V such that $x \in V \subset U$. The system of all neighborhoods of a particular point x is called the neighborhood system of x. One may also consider the ensemble of neighborhood systems for every $x \in X$, which may be called the neighborhood system of the topological space. The natural way to describe this is to define a structure relation on X (cf. Definition 2.51), that contains the pair (x, U) for every point x and every neighborhood U of x. More formally this leads to Definitions 5.74 and 5.76 below. In order to describe the neighborhood system it would be possible to define a function on X such that for each point $x \in X$ the value $f(x)$ is the neighborhood system of x. The domain of such a function is X and its range is a subset of $\mathcal{P}^2(X)$. However, this turns out to be notationally disadvantageous when choosing subsets of the neighborhood system of single points.

Definition 5.74

Let $\xi = (X, \mathcal{T})$ be a topological space and $\mathcal{C}$ the system of all ξ-closed sets. The structure relation $\mathcal{N}_\xi$, or short $\mathcal{N}$, defined by

$$(x, U) \in \mathcal{N} \quad \Longleftrightarrow \quad \exists V \in \mathcal{T} \quad x \in V \subset U$$

is called the **neighborhood system of** ξ. The structure relation

$$\mathcal{N}_\xi^{\text{open}} = \mathcal{N}_\xi \cap (X \times \mathcal{T})$$

is called the **open neighborhood system of** ξ, also denoted by $\mathcal{N}^{\text{open}}$. The structure relation

$$\mathcal{N}_\xi^{\text{closed}} = \mathcal{N}_\xi \cap (X \times \mathcal{C})$$

is called the **closed neighborhood system of** ξ, also denoted by $\mathcal{N}^{\text{closed}}$. ■

Remark 5.75

Given a topological space $(X, \mathcal{T})$, the following statements hold:

(i) $\forall x \in X \quad \forall U \subset X \quad (x, U) \in \mathcal{N}^{\text{open}} \Longleftrightarrow U \in \mathcal{T}(x)$

(ii) $\Phi'\big(\mathcal{N}^{\text{open}}\big) = \Phi'(\mathcal{N}) = \mathcal{N}$

Note that we use the notation of Definition 1.51 in (i) and the map Φ' defined in Definition 5.6 in (ii). ■

We now introduce the neighborhood system of an arbitrary set $A \subset X$. The special case in which A is a singleton leads to the definition of the neighborhood system of a point. Remember that for a relation R on X we have defined

$$\begin{aligned} R\{x\} &= \{y \in X : (x,y) \in R\}, \\ R\langle A\rangle &= \big\{y \in X : \forall x \in A \ \ (x,y) \in R\big\} \end{aligned}$$

Definition 5.76

Let $\mathcal{N}$ be the neighborhood system of a topological space $(X, \mathcal{T})$.

(i) For every $A \subset X$ the system $\mathcal{N}\langle A\rangle$ is called the **neighborhood system of** A, and every member is called a **neighborhood of** A.

(ii) For every $A \subset X$, the systems $\mathcal{N}^{\text{open}}\langle A\rangle$ and $\mathcal{N}^{\text{closed}}\langle A\rangle$ are called the **open** and the **closed neighborhood systems of** A, respectively, and every member is called an **open (closed) neighborhood of** A.

(iii) For every $x \in X$, the system $\mathcal{N}\{x\} = \mathcal{N}\langle\{x\}\rangle$ is called the **neighborhood system of** x. A member $U \in \mathcal{N}\{x\}$ is called **neighborhood of** x.

(iv) For every $x \in X$, the systems

$$\mathcal{N}^{\text{open}}\{x\} = \mathcal{N}^{\text{open}}\langle\{x\}\rangle, \qquad \mathcal{N}^{\text{closed}}\{x\} = \mathcal{N}^{\text{closed}}\langle\{x\}\rangle$$

are called the **open** and **closed neighborhood systems of** x, respectively. Their members are respectively called the **open** and **closed neighborhoods of** x.

■

Notice that for every open $A \subset X$, A is a neighborhoood of itself. Moreover, we have $\mathcal{N}\langle\emptyset\rangle = \mathcal{P}(X)$, i.e. every subset of X is a neighborhood of the empty set. We state a few more consequences of the preceding definitions in the following Lemma.

Lemma 5.77

Let $(X, \mathcal{T})$ be a topological space and $U, V \subset X$. The following statements hold:

(i) $U \in \mathcal{T} \quad \Longleftrightarrow \quad \forall x \in U \quad U \in \mathcal{N}\{x\}$

(ii) $U \in \mathcal{T} \quad \Longleftrightarrow \quad \forall A \subset U \quad U \in \mathcal{N}\langle A\rangle$

(iii) $V \in \mathcal{N}\langle U\rangle \quad \Longleftrightarrow \quad \exists W \in \mathcal{T} \quad U \subset W \subset V$

Proof. To prove (i) assume $U \in \mathcal{N}\{x\}$ holds for every $x \in U$. For each $x \in U$ we may choose $V_x \in \mathcal{T}$ such that $x \in V_x \subset U$. Hence $U = \bigcup_{x \in U} V_x \in \mathcal{T}$. The converse is obvious.
(ii) is a direct consequence of (i).
To prove (iii) assume $V \in \mathcal{N}\langle U\rangle$. Then $V \in \mathcal{N}\{x\}$ for every $x \in U$. For each $x \in U$, we may choose $W_x \in \mathcal{T}$ such that $x \in W_x \subset V$. Therefore $U \subset W \subset V$ where $W = \bigcup_{x \in U} W_x$. The converse is obvious. □

Our definitions of neighborhood and neighborhood system are based on the notion of topological space. In the following Lemma we list properties of the neighborhood system. We then show that for a given structure relation with these properties there is a unique topology such that the neighborhood system is this structure relation.

Lemma 5.78
The neighborhood system $\mathcal{N}$ of a topological space $(X, \mathcal{T})$ has the following properties:

(i) $\forall x \in X \quad \mathcal{N}\{x\} \neq \emptyset$

(ii) $\forall x \in X \quad \forall U \in \mathcal{N}\{x\} \quad x \in U$

(iii) $\forall x \in X \quad \forall U \in \mathcal{N}\{x\} \quad \forall V \supset U \quad V \in \mathcal{N}\{x\}$

(iv) $\forall x \in X \quad \forall U, V \in \mathcal{N}\{x\} \quad U \cap V \in \mathcal{N}\{x\}$

(v) $\forall x \in X \quad \forall U \in \mathcal{N}\{x\} \quad \exists V \in \mathcal{N}\{x\} \quad \forall y \in V \quad U \in \mathcal{N}\{y\}$

Proof. (i) to (iv) are obvious. To show (v), let $x \in X$ and $U \in \mathcal{N}\{x\}$. We may choose $V \in \mathcal{T}$ such that $x \in V \subset U$. Then, for every $y \in V$, we have $U \in \mathcal{N}\{y\}$. □

Lemma 5.79
Let X be a set and $\mathcal{N} \subset X \times \mathcal{P}(X)$ such that properties (i) to (v) in Lemma 5.78 are satisfied. The system $\mathcal{T} = \{U \subset X : \forall x \in U \quad U \in \mathcal{N}\{x\}\}$ is the unique topology on X such that $\mathcal{N}$ is the neighborhood system of $(X, \mathcal{T})$.

Proof. Notice that $\mathcal{T}$ is a topology on X by properties (i), (iii), and (iv). Let $\mathcal{N}_\xi$ be the neighborhood system of $\xi = (X, \mathcal{T})$. We show that $\mathcal{N}_\xi = \mathcal{N}$. Fix $x \in X$. First assume that $U \in \mathcal{N}_\xi\{x\}$. There exists $V \in \mathcal{T}$ such that $x \in V \subset U$. Therefore $V \in \mathcal{N}\{x\}$ by definition of $\mathcal{T}$, and thus $U \in \mathcal{N}\{x\}$.
Now assume $U \in \mathcal{N}\{x\}$. We may define $V = \{y \in X : U \in \mathcal{N}\{y\}\}$. Fix $y \in V$. By property (v) there is $W \in \mathcal{N}\{y\}$ such that $U \in \mathcal{N}\{z\}$ for every $z \in W$. Thus $W \subset V$, and therefore $V \in \mathcal{N}\{y\}$ by property (iii). It follows that $V \in \mathcal{T}$. Since $x \in V \subset U$ by property (ii), we have $U \in \mathcal{N}_\xi\{x\}$.
The uniqueness of $\mathcal{T}$ follows by Lemma 5.77 (i). □

In the following Lemma we list several statements that are satisfied by the neighborhoods of subsets of X—in contrast to the neighborhoods of single points of X considered in Lemma 5.78.

Lemma 5.80
The neighborhood system $\mathcal{N}$ of a topological space $(X, \mathcal{T})$ has the following properties:

(i) $\forall A \subset X \quad \forall U \in \mathcal{N}\langle A \rangle \quad A \subset U$

(ii) $\forall A \subset X \quad \forall U \in \mathcal{N}\langle A \rangle \quad \forall V \supset U \quad V \in \mathcal{N}\langle A \rangle$

(iii) $\forall A \subset X \quad \forall U, V \in \mathcal{N}\langle A \rangle \quad U \cap V \in \mathcal{N}\langle A \rangle$

(iv) $\forall A \subset X \quad \forall U \in \mathcal{N}\langle A \rangle \quad \exists V \in \mathcal{N}\langle A \rangle \quad U \in \mathcal{N}\langle V \rangle$

(v) For every index set I and every $A_i \subset X$ $(i \in I)$, we have

$$\mathcal{N}\Big\langle \bigcup_{i \in I} A_i \Big\rangle = \bigcap_{i \in I} \mathcal{N}\langle A_i \rangle$$

Proof. (i) to (iii) and (v), follow by Lemma 5.77. To show (iv), let $A \subset X$ and $U \in \mathcal{N}\langle A \rangle$. We may choose $V \in \mathcal{T}$ such that $A \subset V \subset U$ by Lemma 5.77 (iii). Then $V \in \mathcal{N}\langle V \rangle$ by Lemma 5.77 (ii), and therefore $U \in \mathcal{N}\langle V \rangle$. □

Lemma 5.81
Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ with $A \neq \emptyset$. The system $\mathcal{N}\langle A \rangle$ is a fixed filter on X.

Proof. It follows by Lemma 5.80 (i) to (iii) that $\mathcal{N}\langle A \rangle$ is a filter. This filter is obviously fixed. □

Notice that in the following Lemma the statements (i) to (v) correspond to the statements (i) to (v) in Lemma 5.80, the statement (vi) guarantees that $\mathcal{M}$ can be derived from a structure relation, and the statement (vii) excludes that the system is empty for any $A \subset X$.

Lemma 5.82
Let X be a set and $\mathcal{M} : \mathcal{P}(X) \longrightarrow \mathcal{P}^2(X)$ a map such that the following statements hold:

(i) $\forall A \subset X \quad \forall U \in \mathcal{M}(A) \quad A \subset U$

(ii) $\forall A \subset X \quad \forall U \in \mathcal{M}(A) \quad \forall V \supset U \quad V \in \mathcal{M}(A)$

(iii) $\forall A \subset X \quad \forall U, V \in \mathcal{M}(A) \quad U \cap V \in \mathcal{M}(A)$

(iv) $\forall A \subset X \quad \forall U \in \mathcal{M}(A) \quad \exists V \in \mathcal{M}(A) \quad U \in \mathcal{M}(V)$

(v) For every index set I and every $A_i \subset X$ $(i \in I)$, we have

$$\mathcal{M}\big(\bigcup_{i\in I} A_i\big) = \bigcap_{i\in I} \mathcal{M}(A_i)$$

(vi) $\mathcal{M}(\emptyset) = \mathcal{P}(X)$

(vii) $\forall A \subset X \quad \mathcal{M}(A) \neq \emptyset$

There is a unique topology $\mathcal{T}$ on X, such that $\mathcal{M}(A) = \mathcal{N}\langle A\rangle$ for every $A \subset X$, where $\mathcal{N}$ is the neighborhood system of $(X, \mathcal{T})$.

Proof. We give two proofs. In the first one we use Lemma 5.79, while in the second one we do not.

First proof:
We may define a relation $\mathcal{N} \subset X \times \mathcal{P}(X)$ by $\mathcal{N}\{x\} = \mathcal{M}(\{x\})$ for every $x \in X$. Then $\mathcal{N}$ has properties (i) to (v) in Lemma 5.78. There is a unique topology $\mathcal{T}$ on X such that $\mathcal{N}$ is the neighborhood system of $(X, \mathcal{T})$ by Lemma 5.79. Fix

$A \subset X$ with $A \neq \emptyset$. Then we have

$$\mathcal{N}\langle A\rangle = \bigcap_{x\in A}\mathcal{N}\{x\} = \bigcap_{x\in A}\mathcal{M}(\{x\}) = \mathcal{M}(A)$$

Moreover $\mathcal{N}\langle\emptyset\rangle = \mathcal{P}(X)$.

Second proof:

We define $\mathcal{T} = \{U \subset X : \forall A \subset U \;\; U \in \mathcal{M}(A)\}$. First we show that $\mathcal{T}$ is a topology on X. Let $U_i \in \mathcal{T}$ $(i \in I)$, where I is an index set, $U = \bigcup_{i\in I} U_i$, and $A \subset U$. Further let $A_i = A \cap U_i$ $(i \in I)$. It follows that $A = \bigcup_{i\in I} A_i$. We have

$$\mathcal{M}(A) = \mathcal{M}\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I}\mathcal{M}(A_i)$$

by (v). Moreover, for every $i \in I$, $A_i \subset U_i$ implies $U_i \in \mathcal{M}(A_i)$, and thus $U \in \mathcal{M}(A_i)$ by (ii). Hence we obtain $U \in \mathcal{M}(A)$. This proves that $\mathcal{T}$ is a topology.

Let $\mathcal{N}$ be the neighborhood system of $\xi = (X, \mathcal{T})$. We show that $\mathcal{N}\langle A\rangle = \mathcal{M}(A)$ for every $A \subset X$. Clearly, $\mathcal{N}\langle\emptyset\rangle = \mathcal{M}(\emptyset)$. Fix $A \subset X$ with $A \neq \emptyset$.

First assume that $U \in \mathcal{N}\langle A\rangle$. There exists $V \in \mathcal{T}$ such that $A \subset V \subset U$. Therefore $V \in \mathcal{M}(A)$ by definition of $\mathcal{T}$, and thus $U \in \mathcal{M}(A)$.

Now assume $U \in \mathcal{M}(A)$, and define $V = \bigcup\{B \subset X : U \in \mathcal{M}(B)\}$. Then we have

$$\begin{aligned}\mathcal{M}(V) &= \mathcal{M}\left(\bigcup\{B \subset X : U \in \mathcal{M}(B)\}\right)\\ &= \bigcap\{\mathcal{M}(B) : B \subset X,\, U \in \mathcal{M}(B)\}\end{aligned}$$

It follows that $U \in \mathcal{M}(V)$. Hence $U \in \mathcal{M}(B)$ for every $B \subset V$ by (v). Fix $B \subset V$. Then there is $W \in \mathcal{M}(B)$ such that $U \in \mathcal{M}(W)$ by (iv). Hence $W \subset V$, and thus $V \in \mathcal{M}(B)$ by (ii). It follows that $V \in \mathcal{T}$. Since $V \subset U$ by (i), we have $U \in \mathcal{N}\langle A\rangle$.

The uniqueness follows by Lemma 5.77 (ii). □

Lemma 5.81 says that the neighboorhood system of a non-empty set $A \subset X$ is a fixed filter. This suggests the idea to define the "neighborhood base" of A as a filter base for that filter. For a point $x \in X$, a neighborhood base is a subset of the neighborhood system of x such that each neighboorhood contains a member of the base. We begin with the definition of neighborhood base, continue with the definition of neighborhood base of a subset of X, and then consider the neighborhood base of a point of X as a special case. It is then shown below that a neigborhood base of a subset is a filter base for the neighborhood system of that subset.

Definition 5.83
Let $\mathcal{N}$ be the neighborhood system of a topological space $\xi = (X, \mathcal{T})$. A system $\mathcal{B} \subset \mathcal{N}$ is called **neighborhood base of** ξ if $\mathcal{N} = \Phi'(\mathcal{B})$ where Φ' is the map defined in Definition 5.6. We also say that $\mathcal{B}$ **generates** $\mathcal{N}$. ■

In general, for a given topology there exists more than one neighborhood base.

Remark 5.84
Given a topological space ξ, the system $\mathcal{N}^{\text{open}}$ is a neighborhood base of ξ by Lemma 5.75 (ii). ■

Definition 5.85
Let $\xi = (X, \mathcal{T})$ be a topological space and $A \subset X$. A system $\mathcal{B} \subset \mathcal{N}\langle A\rangle$ is called **neighborhood base of** A if $\Phi(\mathcal{B}) = \mathcal{N}\langle A\rangle$. If, in addition, $A = \{x\}$ for some $x \in X$, then $\mathcal{B}$ is called **neighborhood base of** x. ■

Again, for a given topology and a given set $A \subset X$, in general, more than one neighborhood base of A exists.

Lemma 5.86
Let $\xi = (X, \mathcal{T})$ be a topological space, $\mathcal{B}$ a neighboorhood base of ξ, and $x \in X$. Then $\mathcal{B}\{x\}$ is a neighborhood base of x.

Proof. This follows by Lemma 5.7 (i). □

Notice that, given a neighborhood base $\mathcal{B}$ of a topological space $(X, \mathcal{T})$ and $A \subset X$, $\mathcal{B}\langle A\rangle$ need not be a neighborhood base of A, because generally equality in Lemma 5.7 (iii) may not hold.

Remark 5.87
Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The following statements hold:

(i) Every neighborhood base of A is a filter base for $\mathcal{N}\langle A\rangle$.

(ii) The system $\mathcal{N}^{\text{open}}\langle A\rangle$ is a neighborhood base of A by Lemma 5.77 (iii).

Notice that $\mathcal{N}^{\text{closed}}\langle A\rangle$ need not be a neighborhood base of A. ■

There is a straightforward way to obtain a neighborhood base of a topological space from a topological base demonstrated in the following Lemma.

Lemma 5.88
Given a topological space $\xi = (X, \mathcal{T})$ and a base $\mathcal{A}$ for $\mathcal{T}$. Then the structure relation $\mathcal{B}$ defined by

$$(x, U) \in \mathcal{B} \iff U \in \mathcal{A}(x)$$

is a neighborhood base of ξ. For every $x \in X$, $\mathcal{A}(x)$ is a neighborhood base of x that has solely open member sets.

Proof. Let $x \in X$ and $V \in \mathcal{N}\{x\}$. We may choose $W \in \mathcal{T}$ such that $x \in W \subset V$, and $U \in \mathcal{A}$ such that $x \in U \subset W$. Then $(x, U) \in \mathcal{B}$. The second claim is obvious.

□

We now address the following two questions: For a given topological space, can a topological base be constructed from a neighborhood base? And: Without specifying a topology beforehand, when is a given structure relation a neighborhood base of some topology?

Lemma 5.89
Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ a neighborhood base that has only open member sets. The system $\mathcal{B}[X] = \bigcup_{x \in X} \mathcal{B}\{x\}$ is a topological base.

Proof. Exercise. □

Concerning the second question we proceed similarly as in the case of neighborhood systems and derive a list of properties of a neighborhood base and then show below that if a structure relation on a set X has these properties, then there is a topology $\mathcal{T}$ on X such that the structure relation is a neighborhood base of $(X, \mathcal{T})$.

Lemma 5.90
Let $\mathcal{B}$ be a neighborhood base of a topological space. $\mathcal{B}$ has the following properties:

(i) $\forall x \in X \quad \mathcal{B}\{x\} \neq 0$

(ii) $\forall x \in X \quad \forall U \in \mathcal{B}\{x\} \quad x \in U$

(iii) $\forall x \in X \quad \forall U, V \in \mathcal{B}\{x\} \quad \exists W \in \mathcal{B}\{x\} \quad W \subset U \cap V$

(iv) $\forall x \in X \quad \forall U \in \mathcal{B}\{x\} \quad \exists V \in \mathcal{B}\{x\} \quad \forall y \in V \quad \exists W \in \mathcal{B}\{y\} \quad W \subset U$

Proof. This follows by Lemma 5.78. □

Lemma 5.91
Let X be a set and $\mathcal{B} \subset X \times \mathcal{P}(X)$ such that properties (i) to (iv) in Lemma 5.90 are satisfied. There is a unique topology $\mathcal{T}$ on X such that $\mathcal{B}$ is a neighborhood base of $(X, \mathcal{T})$.

Proof. The structure relation $\mathcal{N} = \Phi'(\mathcal{B})$ satisfies conditions (i) to (v) in Lemma 5.78. Hence there exists a topology $\mathcal{T}$ on X by Lemma 5.79 such that $\mathcal{N}$ is neighborhood system of $(X, \mathcal{T})$, and thus $\mathcal{B}$ is a neighborhood base of $(X, \mathcal{T})$. □

Recall that a topological space that has a countable base is called second countable. The following definition refers to the cardinality of the neighborhood base of every point of X.

Definition 5.92
Let $\xi = (X, \mathcal{T})$ be a topological space. If there is a neighborhood base $\mathcal{B}$ of ξ such that $\mathcal{B}\{x\}$ is countable for every $x \in X$, then the space ξ or the topology $\mathcal{T}$ is called **first countable**. ■

Example 5.93
Given a set X, the discrete topology $\mathcal{T}_{\text{dis}}$ on X, and a point $x \in X$, the neighborhood system of x is $\mathcal{T}_{\text{dis}}(x)$. $\mathcal{T}_{\text{dis}}$ is clearly first countable. ■

Example 5.94
Given a set X, the indiscrete topology $\mathcal{T}_{\text{in}}$ on X, and a point $x \in X$, the neighborhood system of x is $\{X\}$. ■

Lemma 5.95
A second countable topological space $(X, \mathcal{T})$ is also first countable.

Proof. This follows by Lemma 5.88. □

Lemma 5.96
Let $(X, \mathcal{T})$ be a first countable topological space and $x \in X$. There is a neighborhood base $\{B_m : m \in \mathbb{N}\}$ of x such that all B_m $(m \in \mathbb{N})$ are open and

$$\forall m, n \in \mathbb{N} \quad m < n \implies B_n \subset B_m$$

Proof. By Lemma 5.88, we may choose a neighborhood base $\{C_m : m \in \mathbb{N}\}$ for x such that all C_m $(m \in \mathbb{N})$ are open. For every $n \in \mathbb{N}$ we define $B_n = \bigcap \{C_m : m \in \mathbb{N}, m \leq n\}$. □

We conclude this section with a criterion for the comparison of two topologies on the same set in terms of neighborhoods and neighborhood bases.

Lemma 5.97
Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X, $\mathcal{N}_1$ and $\mathcal{N}_2$ their respective neighborhood systems, and $\mathcal{B}_1$ and $\mathcal{B}_2$ two neighborhood bases generating $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. The following three statements are equivalent:

(i) $\mathcal{T}_2 \subset \mathcal{T}_1$

(ii) $\forall x \in X \quad \mathcal{N}_2\{x\} \subset \mathcal{N}_1\{x\}$

(iii) $\forall x \in X \quad \mathcal{B}_2\{x\} \subset_\Phi \mathcal{B}_1\{x\}$

Proof. Since $\mathcal{B}_1\{x\}$ is a filter base for $\mathcal{N}_1\{x\}$, and $\mathcal{B}_2\{x\}$ is a filter base for $\mathcal{N}_2\{x\}$ by Remark 5.87 (i), statements (ii) and (iii) are equivalent by Lemma 5.59 (i). The fact that (i) implies (ii) follows by the definition of neighborhood system. The converse implication follows by Lemma 5.77 (i). □

5.5 Interval topology

In this and in the next Section we discuss two specific kinds of topologies: those associated to pre-orderings with full field, and those generated by pseudo-metrics.

Lemma and Definition 5.98
Given a pre-ordered space (X, R) where R has full field, the system

$$\mathcal{S} = \big\{ \,]{-\infty}, x[\;,\;]x, \infty[\; : \, x \in X \big\} \cup \{\emptyset\}$$

is a topological subbase on X. The generated topology is called R-**interval topology**, or short **interval topology**, and also written $\tau(R)$. If $Y \subset X$ is order dense, then also the system

$$\mathcal{R} = \big\{ \,]{-\infty}, y[\;,\;]y, \infty[\; : \, y \in Y \big\} \cup \{\emptyset\}$$

is a subbase for the interval topology.

Proof. We first show that $\mathcal{S}$ is a topological subbase by Lemma 5.38. Since R has full field, there is, for each $x \in X$, a point $y \in X$ such that $x \in\,]{-\infty}, y[$ or $x \in\,]y, \infty[$. It follows that $\bigcup \mathcal{S} = X$, and clearly $\mathcal{S} \neq \emptyset$.
If Y is order dense, then $\mathcal{S} \subset \Theta(\mathcal{R})$ by Remark 2.31, and therefore $\bigcup \mathcal{R} = X$. Thus $\mathcal{R}$ is a topological subbase. Moreover, $\Psi(\mathcal{S}) \subset \Psi\,\Theta\,(\mathcal{R}) \subset \Theta\,\Psi\,(\mathcal{R})$ by Lemma 5.4, and hence $\Theta\,\Psi\,(\mathcal{S}) \subset \Theta\,\Psi\,(\mathcal{R})$. Therefore $\mathcal{R}$ and $\mathcal{S}$ generate the same topology. □

Notice that for some pre-ordered spaces $(X, \prec)$ the system

$$\mathcal{A} = \big\{ \,]{-\infty}, x[\;,\;]x, \infty[\; : \, x \in X \big\}$$

may contain $\emptyset$ or may not have the finite intersection property. In these two cases $\mathcal{A}$ alone (without explicitly including $\emptyset$) is a topological subbase.

Under additional assumptions on the pre-ordered space, canonical bases for the interval topology can be specified.

Lemma 5.99
Let $(X, \prec)$ be a pre-ordered space where $\prec$ has full field and the systems

$$\mathcal{S}_- = \big\{\,]{-\infty}, x[\;:\; x \in X \big\} \cup \{\emptyset\}\,, \qquad \mathcal{S}_+ = \big\{\,]x, \infty[\;:\; x \in X \big\} \cup \{\emptyset\}$$

satisfy $\Psi(\mathcal{S}_-) = \mathcal{S}_-$ and $\Psi(\mathcal{S}_+) = \mathcal{S}_+$, that is $\mathcal{S}_-$ and $\mathcal{S}_+$ are fixed points of Ψ. Further let

$$\mathcal{S} \;=\; \mathcal{S}_+ \cup \mathcal{S}_-\,, \qquad \mathcal{A} = \big\{\,]x, y[\;:\; x, y \in X,\; x \prec y \big\} \cup \{\emptyset\}$$

The system $\mathcal{A} \cup \mathcal{S}$ is a base for the interval topology. If $\prec$ has full domain and full range, then $\mathcal{A}$ alone is a base for the interval topology.
Furthermore, let $Y \subset X$ be an order dense set such that the systems

$$\mathcal{R}_- = \big\{\,]{-\infty}, x[\;:\; x \in Y \big\} \cup \{\emptyset\}\,, \qquad \mathcal{R}_+ = \big\{\,]x, \infty[\;:\; x \in Y \big\} \cup \{\emptyset\}$$

satisfy $\Psi(\mathcal{R}_-) = \mathcal{R}_-$ and $\Psi(\mathcal{R}_+) = \mathcal{R}_+$. Moreover, let

$$\mathcal{R} \;=\; \mathcal{R}_- \cup \mathcal{R}_+\,, \qquad \mathcal{B} = \big\{\,]x, y[\;:\; x, y \in Y,\; x \prec y \big\} \cup \{\emptyset\}$$

Then $\mathcal{B} \cup \mathcal{R}$ is a base for the interval topology. If $\prec$ has full domain and full range, then $\mathcal{B}$ alone is a base for the interval topology.

Proof. To prove that $\mathcal{A} \cup \mathcal{S}$ is a base for the interval topology, we show that $\Psi(\mathcal{S}) = \mathcal{A} \cup \mathcal{S}$. Let $\emptyset \neq A \in \Psi(\mathcal{S})$. We have

$$(A = A_E) \;\vee\; (A = A_F) \;\vee\; (A = A_E \cap A_F)$$

where

$$A_E \;=\; \bigcap \big\{\,]x, \infty[\;:\; x \in E \big\}, \quad A_F \;=\; \bigcap \big\{\,]{-\infty}, y[\;:\; y \in F \big\}$$

and $E, F \sqsubset X$ are index sets. Since $\mathcal{S}_-$ and $\mathcal{S}_+$ are fixed points of Ψ, there are points $x, y \in X$ such that $A_E = \,]x, \infty[$ and $A_F = \,]-\infty, y[$.
To prove the second claim we show that $\Theta\,\Psi(\mathcal{S}) = \Theta(\mathcal{A})$ under the given conditions. We may choose A, A_E, and A_F as above. Since $\prec$ has full domain, we have

$$A_E \;=\;]x, \infty[\;=\; \bigcup \{\,]x, z[\;:\; z \in X,\; x \prec z \}$$

by Remark 2.28. Thus, if $A = A_E$, then $A \in \Theta(\mathcal{A})$. Similarly, since $\prec$ has full range, we have

$$A_F \;=\;]-\infty, y[\;=\; \bigcup \{\,]z, y[\;:\; z \in X,\; z \prec y \}$$

Hence, if $A = A_F$, then $A \in \Theta(\mathcal{A})$. If $A = A_E \cap A_F$, we have $A = \,]x, y[$ and therefore $A \in \Theta(\mathcal{A})$.
To see that $\mathcal{B} \cup \mathcal{R}$ is a base for the interval topology, we show that $\Psi(\mathcal{R}) = \mathcal{B} \cup \mathcal{R}$. Let $\emptyset \neq B \in \Psi(\mathcal{R})$. Since $\mathcal{R}_-$ and $\mathcal{R}_+$ are fixed points of Ψ, we have either $B = \,]u, \infty[$ or $B = \,]-\infty, v[$ or $B = \,]u, v[$ where $u, v \in Y$.
The last claim follows by Remarks 2.28 and 2.31. □

Note that in the particular case in which $\prec$ is a connective pre-ordering each of the systems $\mathcal{S}_-$, $\mathcal{S}_+$, $\mathcal{R}_-$, and $\mathcal{R}_+$ is a fixed point of Ψ by Lemma 3.60.
In order to handle conditions such as those of Lemma 5.99 in a stringent manner, we define the following notions.

Definition 5.100
Let $(X, \prec)$ be a pre-ordered space. If the systems

$$\mathcal{S}_- = \{\,]-\infty, x[\;:\; x \in X \} \cup \{\emptyset\}\,, \qquad \mathcal{S}_+ = \{\,]x, \infty[\;:\; x \in X \} \cup \{\emptyset\}$$

are fixed points of Ψ, we say that $\prec$ **has the interval intersection property**. If, in addition, $\prec$ has full domain and full range, then $\prec$ is called **interval relation**. ■

Definition 5.101

The interval topology of $(\mathbb{R}, <)$, where $<$ is the standard ordering in the sense of "$<$" on $\mathbb{R}$, is called **standard topology on** $\mathbb{R}$. The interval topology on $(\mathbb{R}_+, <)$ is called **standard topology on** $\mathbb{R}_+$. ■

Remark 5.102

The system $\big\{]-\infty, x[\ ,]x, \infty[\ :\ x \in \mathbb{R} \big\}$ is a subbase for the standard topology on $\mathbb{R}$. The ordering $<$ is an interval relation on $\mathbb{R}$. Thus the system

$$\big\{]x, y[\ :\ x, y \in \mathbb{R},\ x < y \big\} \cup \{\emptyset\}$$

is a base for the standard topology by Lemma 5.99. Moreover, the system

$$\big\{]x, y[\ :\ x, y \in \mathbb{D},\ x < y \big\} \cup \{\emptyset\}$$

is a base for the same topology by Lemmas 4.46 and 5.99. Hence the standard topology on $\mathbb{R}$ is second countable by Remark 4.45 and Lemma 3.69. ■

Remark 5.103
The system

$$\mathcal{S} = \big\{\,]-\infty, x[\,,\ ]x, \infty[\ : x \in \mathbb{R}_+ \big\}$$

is a subbase for the standard topology on $\mathbb{R}_+$. Let

$$\mathcal{A} = \big\{\,]x, y[\ : x, y \in \mathbb{R}_+,\ x < y \big\} \cup \{\emptyset\}$$

The system $\mathcal{S} \cup \mathcal{A}$ is a base for the standard topology by Lemma 5.99. The relation $<$ has the interval intersection property. However it is not an interval relation because it has a minimum. Further let

$$\mathcal{R} = \big\{\,]-\infty, x[\,,\ ]x, \infty[\ : x \in \mathbb{D}_+ \big\}, \quad \mathcal{B} = \big\{\,]x, y[\ : x, y \in \mathbb{D}_+,\ x < y \big\} \cup \{\emptyset\}$$

The system $\mathcal{R} \cup \mathcal{B}$ is a base for the same topology by Lemmas 4.15 and 5.99. Hence the standard topology on $\mathbb{R}_+$ is second countable by Corollary 4.3 and Lemmas 3.69 and 3.70 . ■

We now generalize the concept of interval topology to the case of a system of pre-orderings on the same set X.

Lemma and Definition 5.104
Let X be a set and $\mathcal{R}$ a system of pre-orderings on X where $\mathcal{R}$ has full field. Then the system

$$\mathcal{S} = \big\{ \,]{-\infty}, x[_R\,, \;]x, \infty[_R \,:\, x \in X,\, R \in \mathcal{R} \big\} \cup \{\emptyset\}$$

is a topological subbase. The topology generated by $\mathcal{S}$ is called **$\mathcal{R}$-interval topology**, and written $\tau(\mathcal{R})$. Further let $Y \subset X$ be $\mathcal{R}$-dense in X. Then

$$\mathcal{Q} = \big\{ \,]{-\infty}, y[_R\,, \;]y, \infty[_R \,:\, y \in Y,\, R \in \mathcal{R} \big\} \cup \{\emptyset\}$$

is a subbase for the same topology.

Proof. To see that $\mathcal{S}$ is a topological subbase, note that for every $x \in X$ there is $R \in \mathcal{R}$ and $y \in X$ such that $x \in \;]{-\infty}, y[_R$ or $x \in \;]y, \infty[_R$. It follows that $\bigcup \mathcal{S} = X$, and clearly $\mathcal{S} \neq \emptyset$.
To see the second claim, assume that Y is $\mathcal{R}$-dense. Then $\mathcal{S} \subset \Theta(\mathcal{Q})$ by Remark 2.31, and therefore $\bigcup \mathcal{Q} = X$. Thus $\mathcal{Q}$ is a topological subbase. Moreover, $\Psi(\mathcal{S}) \subset \Psi\,\Theta\,(\mathcal{Q}) \subset \Theta\,\Psi\,(\mathcal{Q})$ by Lemma 5.4, and hence $\Theta\,\Psi\,(\mathcal{S}) \subset \Theta\,\Psi\,(\mathcal{Q})$. Therefore $\mathcal{Q}$ and $\mathcal{S}$ generate the same topology. □

Lemma 5.105
Let X be a set and $\mathcal{R}$ a system of pre-orderings on X where $\mathcal{R}$ is independent and has full field. Further let $S = \bigcap \mathcal{R}$. Then we have $\tau(\mathcal{R}) \subset \tau(S)$. If $\mathcal{R}$ is finite, then $\tau(S) \subset \tau(\mathcal{R})$

Proof. This follows by Lemma 2.87. □

Example 4.51 is an important example of the construction defined in Definition 5.104. We introduce the following notion.

Definition 5.106
Let $n \in \mathbb{N}$, $n \geq 1$, and $(X_i, R_i) = (\mathbb{R}, <)$ for every $i \in \mathbb{N}$, $1 \leq i \leq n$. Further let $X = \bigtimes_{i=1}^{n} X_i$ and $\mathcal{R} = \left\{p_i^{-1}[R_i] \,:\, i \in \mathbb{N},\, 1 \leq i \leq n\right\}$. The $\mathcal{R}$-interval topology on $\mathbb{R}^n$ is called **standard topology on** $\mathbb{R}^n$. ■

Remark 5.107
With definitions as in Definition 5.106, the system

$$\mathcal{S} = \left\{\,]{-\infty}, x[_R\,,\ ]x, \infty[_R \,:\, x \in \mathbb{D}^n,\, R \in \mathcal{R}\right\}$$

is a subbase for the standard topology on $\mathbb{R}^n$ by Lemma 5.104. Since $\Psi(\mathcal{S})$ is a base, this topology is second countable by Remark 4.45 and Lemmas 3.69, 3.70, and 3.71. Let $S = \bigcap \mathcal{R}$. Since $\mathcal{R}$ is independent, the S-interval topology is the standard topology on $\mathbb{R}^n$ by Lemma 5.105. Since S is an interval relation, the system

$$\left\{\,]x, y[\,:\, x, y \in \mathbb{R}^n,\, (x, y) \in S\right\} \cup \{\emptyset\},$$

where the interval refers to the ordering S, is a base for this topology by Lemma 5.99. Similarly, it is clear that the system

$$\left\{\,]x, y[\,:\, x, y \in \mathbb{D}^n,\, (x, y) \in S\right\} \cup \{\emptyset\}$$

is a base for the same topology. ■

Similarly, a standard topology on any finite product of positive reals can be defined as follows.

Definition 5.108
Let $n \in \mathbb{N}$, $n \geq 1$, and $(X_i, R_i) = (\mathbb{R}_+, <)$ for every $i \in \mathbb{N}$, $1 \leq i \leq n$. Further let $\mathcal{R} = \left\{p_i^{-1}[R_i] \,:\, i \in \mathbb{N},\, 1 \leq i \leq n\right\}$. The $\mathcal{R}$-interval topology on $\mathbb{R}_+^n$ is called **standard topology on** $\mathbb{R}_+^n$. ■

Remark 5.109
With definitions as in Definition 5.108, the system

$$\mathcal{S} = \left\{ \left]-\infty, x\right[_R , \left]x, \infty\right[_R : x \in \mathbb{D}^n_+,\ R \in \mathcal{R} \right\}$$

is a subbase for the standard topology on $\mathbb{R}^n_+$ by Lemma 5.104. Since $\Psi(\mathcal{S})$ is a base, this topology is second countable by Corollary 4.3 and Lemmas 3.69, 3.70, and 3.71. $\mathcal{R}$ is upwards independent, but not downwards independent. ■

It is possible to construct other topologies on a pre-ordered space. While using all improper intervals and the empty set as a subbase in Lemma and Definition 5.98, we may use only the lower or only the upper segments, respectively supplemented by the empty set, as subbase as done in the following Lemma.

Lemma 5.110
Let $(X, \prec)$ be a pre-ordered space. If $\prec$ has full domain, then the system

$$\mathcal{S}_- = \left\{ \,]-\infty, x[\; : \; x \in X \right\} \cup \{\emptyset\}$$

is a topological subbase on X. If $\prec$ has full range, then the system

$$\mathcal{S}_+ = \left\{ \,]x, \infty[\; : \; x \in X \right\} \cup \{\emptyset\}$$

is a topological subbase on X.
Let $Y \subset X$ be order dense. If $\prec$ has full domain, then the system

$$\mathcal{R}_- = \left\{ \,]-\infty, y[\; : \; y \in Y \right\} \cup \{\emptyset\}$$

is a topological subbase on X. If $\prec$ has full range, then the system

$$\mathcal{R}_+ = \left\{ \,]y, \infty[\; : \; y \in Y \right\} \cup \{\emptyset\}$$

is a topological subbase on X.

Proof. Exercise. □

Clearly, if a topological subbase is a fixed point of Ψ, it is a base for its generated topology. In case of the interval topology it is, under certain conditions, even a topology if we only include X to the set system.

Lemma 5.111

Let $(X, \prec)$ be a totally ordered space in the sense of "$<$". We define the systems

$$\mathcal{S}_- = \big\{ \,]-\infty, x[\; : \; x \in X \big\} \cup \{\emptyset, X\} \,,$$
$$\mathcal{S}_+ = \big\{ \,]x, \infty[\; : \; x \in X \big\} \cup \{\emptyset, X\}$$

If $\prec$ has the least upper bound property, then $\Theta(\mathcal{S}_-) = \mathcal{S}_-$ and $\Theta(\mathcal{S}_+) = \mathcal{S}_+$. In this case, $\mathcal{S}_-$ and $\mathcal{S}_+$ are topologies on X.

Proof. Let I be an index set, $x_i \in X$ $(i \in I)$, and $A = \{x_i \, : \, i \in I\}$. If A has no upper bound, then we have $X = \bigcup_{i \in I} \;]-\infty, x_i[$. If A has an upper bound, let x be the supremum of A. Then we have $]-\infty, x[\; = \bigcup_{i \in I} \;]-\infty, x_i[$. It follows that $\Theta(\mathcal{S}_-) = \mathcal{S}_-$. The second equation follows similarly by Theorem 2.49. Since $\mathcal{S}_-$ and $\mathcal{S}_+$ are fixed points of Ψ, the last claim follows. □

Example 5.112

Let $<$ denote the standard ordering in the sense of "$<$" on $\mathbb{R}$. The system

$$\mathcal{T}_< = \big\{ \,]-\infty, x[\; : \; x \in \mathbb{R} \big\} \cup \{\emptyset, X\}$$

is a topology on $\mathbb{R}$. The system

$$\big\{ \,]-\infty, x[\; : \; x \in \mathbb{D} \big\} \cup \{\emptyset\}$$

is a base for this topology. ■

Example 5.113
Let $\leq$ denote the standard ordering in the sense of "$\leq$" on $\mathbb{R}$. Notice that we have $x \in \,]-\infty, x[$ for every $x \in \mathbb{R}$. The system

$$\big\{ \,]-\infty, x[\; : \; x \in \mathbb{R} \big\} \cup \{\emptyset\}$$

is a topological base. However, the system

$$\big\{ \,]-\infty, x[\; : \; x \in \mathbb{D} \big\} \cup \{\emptyset\}$$

is not a base for the generated topology. The conditions of Lemma 5.110 are not satisfied as $\mathbb{D}$ is not order dense. ■

5.6 Pseudo-metrics

In this Section we consider topologies generated by pseudo-metrics and metrics. Some of the most important examples of metric spaces are the real numbers $\mathbb{R}$ and their n-fold Cartesian product $\mathbb{R}^n$ together with the Euclidean metric. This metric is only introduced in Chapter 8 as it requires the definition of the square root function on the reals. The concept of pseudo-metric space can be generalized to that of uniform space, which is a set together with a system of—generally more than one—pseudo-metrics.
We start with the basic definitions.

Definition 5.114
Given a set X, a **pseudo-metric on** X is a map $d : X \times X \longrightarrow \mathbb{R}_+$ with the following properties:

(i) $\forall x \in X \quad d(x,x) = 0$

(ii) $\forall x, y \in X \quad d(y,x) = d(x,y) \quad$ (symmetry)

(iii) $\forall x, y, z \in X \quad d(x,z) \leq d(x,y) + d(y,z) \quad$ (triangle inequality)

The pair (X, d) is called **pseudo-metric space**. The map d is called **metric on** X if the following statement holds instead of (i):

(i)$'$ $\forall x, y \in X \quad x = y \iff d(x,y) = 0$

In this case (X, d) is called **metric space**. ■

Definition 5.115
Given a pseudo-metric space (X, d), d is called **bounded** if there is $r \in \mathbb{R}_+$ such that

$$\forall x, y \in X \quad d(x,y) < r$$

■

Remark 5.116
Given a set X and a bounded pseudo-metric d on X, we have

$$\sup\left\{d(x,y) \,:\, x, y \in X\right\} < \infty$$

by Lemma 4.39. ■

Example 5.117
Let X be a set and (Y, d) a bounded pseudo-metric space. The map

$$D : Y^X \times Y^X \longrightarrow \mathbb{R}_+ , \qquad D(f, g) = \sup_{x \in X} d\big(f(x), g(x)\big)$$

is a bounded pseudo-metric on Y^X. If d is a metric, then D is a metric. ■

Definition 5.118
Let (X, d_X) and (Y, d_Y) be pseudo-metric spaces and $f : X \longrightarrow Y$ a surjective map. f is called **isometry** if $d_Y(f(x), f(y)) = d_X(x, y)$ for every $x, y \in X$. ■

In the following it is shown how a topology can be related to a given pseudo-metric on a set X. In this sense, a pseudo-metric space is a special case of a topological space.

Lemma and Definition 5.119
Let (X, d) be a pseudo-metric space. We define the function

$$B : X \times \,]0, \infty[\; \longrightarrow \mathcal{P}(X), \quad B(x, r) = \{y \in X \,:\, d(x, y) < r\}$$

Further let $R \subset \,]0, \infty[$ such that for every $K \in \,]0, \infty[$ there exists $r \in R$ with $r < K$. The system

$$\mathcal{B} \;=\; \big\{B(x, r) \,:\, x \in X,\; r \in R\big\} \cup \{\emptyset\}$$

is a topological base on X. The topology $\Theta(\mathcal{B})$ is called **pseudo-metric topology of** (X, d). This topology is also denoted by $\tau(d)$. The definition of $\tau(d)$ is independent of the choice of R. We say that d **generates** $\tau(d)$.
If d is a metric, then $\tau(d)$ is also called **metric topology**. For every $r \in \,]0, \infty[$ and $x \in X$, the set $B(x, r)$ is $\tau(d)$-open. It is called **open sphere about** x **with** d**-radius** r. For every $r \in \,]0, \infty[$ and $x \in X$, the set $\{y \in X \,:\, d(x, y) \leq r\}$ is $\tau(d)$-closed. It is called **closed sphere about** x **with** d**-radius** r. Moreover, the system

$$\mathcal{C} = \big\{\big(x, B(x, r)\big) \,:\, x \in X,\, r \in R\big\}$$

is a neighborhood base of $(X, \tau(d))$.

Proof. The claims that $\mathcal{B}$ is a topological base on X and that closed spheres are closed sets follow by Lemma and Definition 5.24 and the triangle inequality (exercise).
Now it is clear that $\mathcal{C}$ is a neighborhood base.
To see that $\tau(d)$ is independent of the choice of the radii, let, for $i \in \{1, 2\}$, $R_i \subset \,]0, \infty[$ such that for every $K \in \,]0, \infty[$ there exists $r \in R_i$ with $r < K$. We denote by $\mathcal{T}_i$ $(i \in \{1, 2\})$ the respective pseudo-metric topologies. For $i \in \{1, 2\}$, the system

$$\mathcal{C}_i = \big\{\big(x, B(x, r)\big) \,:\, x \in X,\, r \in R_i\big\}$$

is a neighborhood base of $(X, \mathcal{T}_i)$. For every $x \in X$ we have $\mathcal{C}_2\{x\} \subset_\Phi \mathcal{C}_1\{x\}$ and $\mathcal{C}_1\{x\} \subset_\Phi \mathcal{C}_2\{x\}$. It follows by Lemma 5.97 that $\mathcal{T}_1 = \mathcal{T}_2$. □

Note that we may choose $R = \,]0, \infty[\,$ in Lemma and Definition 5.119.

Example 5.120

Let X be a set and $d : X \times X \longrightarrow \mathbb{R}_+$ a map with $d(x, x) = 0$ for every $x \in X$, and $d(x, y) = 1$ for every $x, y \in X$ with $x \neq y$. Then d is a bounded metric, and $\tau(d)$ is the discrete topology on X. ■

Example 5.121

Let X be a set and $d : X \times X \longrightarrow \mathbb{R}_+$ a map with $d(x, y) = 0$ for every $x, y \in X$. Then d is a bounded pseudo-metric, and $\tau(d)$ is the indiscrete topology on X. ■

Lemma 5.122

The function $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+$, $d(x, y) = |x - y|$, is a metric. The generated topology $\tau(d)$ is the standard topology on $\mathbb{R}$.

Proof. Exercise. □

Lemma and Definition 5.123

Let $n \in \mathbb{N}$ with $n \geq 1$. The function

$$d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_+ , \quad d(x, y) = \max\left\{|x_k - y_k| \;:\; k \in \mathbb{N},\, 1 \leq k \leq n\right\}$$

is a metric. It is called **maximum metric on** $\mathbb{R}^n$.

Proof. Exercise. □

It is shown in Chapter 8 that the topology generated by the maximum metric is the standard topology on $\mathbb{R}^n$.
Given a topological space one may raise the question whether there exists a metric or pseudo-metric that generates it.

Definition 5.124
Given a topological space $\xi = (X, \mathcal{T})$, ξ is called **pseudo-metrizable** if there exists a pseudo-metric d on X such that $\tau(d) = \mathcal{T}$. Similarly, ξ is called **metrizable** if there exists a metric d on X such that $\tau(d) = \mathcal{T}$. ■

Given the fact that pseudo-metric and metric spaces are very important mathematical concepts, pseudo-metrizability and metrizability are central features a topological space may or may not have.
The following Lemma provides a tool to compare the topologies generated by two different pseudo-metrics on the same set X. In particular, it can be used to prove that two specific different pseudo-metrics generate the same topology.

Lemma 5.125
Let X be a set. For $i \in \{1, 2\}$ let d_i be a pseudo-metric on X and define the function

$$B_i : X \times \left]0, \infty\right[\longrightarrow \mathcal{P}(X), \quad B_i(x, r) = \{y \in X \,:\, d_i(x, y) < r\}$$

That is, for $i \in \{1, 2\}$, $x \in X$, and $r \in \left]0, \infty\right[$, the set $B_i(x, r)$ is the open sphere about x with d_i-radius r. $\tau(d_1)$ is finer than $\tau(d_2)$ iff for every $x \in X$ and $r \in \left]0, \infty\right[$, there is $s \in \left]0, \infty\right[$ such that $B_1(x, s) \subset B_2(x, r)$.

Proof. This follows by Lemma 5.97. □

The following Corollary is an application of Lemma 5.125. It shows that for every pseudo-metric space there is a pseudo-metric bounded by 1 that generates the same topology as the original pseudo-metric.

Corollary 5.126

Let (X, d) be a pseudo-metric space. We define the functions $e : X \times X \longrightarrow \mathbb{R}_+$ where $e(x, y)$ is the minimum of $d(x, y)$ and 1, and

$$f : X \times X \longrightarrow \mathbb{R}_+, \quad f(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Both e and f are pseudo-metrics on X. All three pseudo-metrics generate the same topology. Further, if d is a metric, then each of the maps e and f is a metric, too.

Proof. Exercise. □

We finally obtain the following result about pseudo-metric topologies.

Theorem 5.127

For every pseudo-metric space the generated topology is first countable.

Proof. We may choose $R = \mathbb{D}_+ \backslash \{0\}$ in Lemma and Definition 5.119. Then R is countable by Corollary 4.3. Hence there is a countable neighborhood base of x for every $x \in X$ by Lemma and Definition 5.119. □

Chapter 6

Convergence and continuity

This Chapter is devoted to the notion of convergence, and to the related topics of limit points, adherence points, continuity of functions, as well as the closure, interior, and boundary of sets. There are three different concepts of convergence, which are interrelated: sequences, nets, and filters. Each of these concepts is treated in one of the following Sections.

6.1 Sequences

The elementary concept of convergence uses sequences as defined in the following Definition.

Definition 6.1
Given a set X, a **sequence in** X is a function $x : \mathbb{N} \longrightarrow X$. It is denoted by $(x_n : n \in \mathbb{N})$, or, if it is known from the context that x is a sequence, by the short notations (x_n) or x_n. The value $x(n)$ of x at a point $n \in \mathbb{N}$ is also denoted by x_n. ■

In Chapter 5 we have introduced topological spaces and pseudo-metric spaces. In both cases we may define, using sequences, the notions of convergence, limit points, and adherence points. It is then demonstrated below that the respective definitions agree for a pseudo-metric space and the generated pseudo-metric topology.

Definition 6.2
Let X be a set, (x_n) a sequence in X, and $\varphi(x)$ a formula. We say that $\varphi(x_n)$ is true **eventually**, or short $\varphi(x_n)$ **eventually**, if there is $N \in \mathbb{N}$ such that $\varphi(x_n)$ is true for every $n \in \mathbb{N}$ with $N \leq n$. We say that $\varphi(x_n)$ is true **frequently**, or short $\varphi(x_n)$ **frequently**, if, for every $N \in \mathbb{N}$, there is $n \in \mathbb{N}$, $N \leq n$ such that $\varphi(x_n)$ is true. ■

Notice that Definition 6.2 is actually not a single definition but provides one definition for each formula $\varphi(x)$. In the remainder of the text the notions "eventually" and "frequently" are, however, always used together with specific formulae, not with generic formula variables. Thus Definition 6.2 should be regarded as an abbreviated form of writing down a finite number of Definitions. This issue is similar to the one mentioned in the context of the Separation schema 1.4 and the Replacement schema 1.47, see the discussion below Axiom 1.4.

Definition 6.3
Given a topological space $(X, \mathcal{T})$, a subset $A \subset X$, and a sequence (x_n) in A, a point $x \in X$ is called **limit point of** (x_n) if $x_n \in U$ eventually for every $U \in \mathcal{N}\{x\}$. In this case we say that (x_n) **converges** to x, or write $x_n \to x$. The set of all limit points of (x_n) is denoted by $\lim_n x_n$. If (x_n) has a limit point, it is called **convergent**. If (x_n) has a unique limit point, say x, we have $\lim_n x_n = \{x\}$, and we also write $\lim_n x_n = x$. Further, we say that a point $x \in X$ is an **adherence point of** (x_n) if $x_n \in U$ frequently for every $U \in \mathcal{N}\{x\}$. The set of all adherence points of (x_n) is denoted by $\operatorname{adh}_n x_n$. If (x_n) has a unique adherence point, say x, we have $\operatorname{adh}_n x_n = \{x\}$, and we also write $\operatorname{adh}_n x_n = x$.
■

Definition 6.4
Given a pseudo-metric space (X, d), a subset $A \subset X$, and a sequence (x_n) in A, a point $x \in X$ is called **limit point of** (x_n) if $d(x, x_n) < \varepsilon$ eventually for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. In this case we say that (x_n) **converges** to x, or write $x_n \to x$. Further we say that a point $x \in X$ is an **adherence point of** (x_n) if $d(x, x_n) < \varepsilon$ frequently for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. For the limit points and for the adherence points of (x_n) we use the same notations as for limit points and adherence points of sequences in a topological space. ■

Remark 6.5
Let (X, d) be a pseudo-metric space, (x_n) a sequence in X, and $x \in X$. Since $\mathbb{D}_+$ is dense in $\mathbb{R}_+$, we have:

(i) $x \in \lim_n x_n \iff \big(d(x, x_n) < \varepsilon$ eventually for every $\varepsilon \in \mathbb{D}_+ \backslash \{0\} \big)$

(ii) $x \in \operatorname{adh}_n x_n \iff \big(d(x, x_n) < \varepsilon$ frequently for every $\varepsilon \in \mathbb{D}_+ \backslash \{0\} \big)$

■

Obviously a limit point of a sequence in a topological space or in a pseudo-metric space is also an adherence point of the sequence. In general, a sequence (x_n) can have more than one adherence point. A sequence can also have more than one limit point.

Lemma 6.6
Given a topological space $(X, \mathcal{T})$ and a sequence (x_n) in X, a point $x \in X$ is a limit point of (x_n) iff there is a neighborhood base $\mathcal{B}$ of x such that $x_n \in B$ eventually for every $B \in \mathcal{B}$.

Proof. Exercise. □

The following Lemma ensures that the repeated definition of the same notions in Definitions 6.3 and 6.4 is meaningful.

Lemma 6.7

Let (X, d) be a pseudo-metric space, (x_n) a sequence in X, and $x \in X$. Then (x_n) converges to x with respect to d in the sense of Definition 6.4 iff it converges to x with respect to $\tau(d)$ in the sense of Definition 6.3.

Proof. Exercise. □

Definition 6.8

Let X and Y be two sets, (x_n) a sequence in X, and (y_n) a sequence in Y. If there exists a strictly increasing map $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $y_n = x_{f(n)}$ for every $n \in \mathbb{N}$, then (y_n) is called **subsequence of** (x_n), or short $(y_n) \subset (x_n)$. ■

Theorem 6.9

Given a pseudo-metric space (X, d) and a sequence (x_n) in X, x is adherence point of (x_n) iff there exists a subsequence $(y_n) \subset (x_n)$ such that $y_n \to x$.

Proof. Assume $x \in \operatorname{adh}_n x_n$. We may choose a bijection $f : \mathbb{N} \longrightarrow \mathbb{D}_+ \setminus \{0\}$ by Corollary 4.3. We define a map $g : \mathbb{N} \longrightarrow \mathbb{N}$ where $g(m)$ is the minimum of

$$\big\{k \in \mathbb{N} \,:\, k \geq m,\, d(x, x_k) < e\big\}$$

and e is the minimum of the finite set $f\,[\sigma(m)]$. g is clearly unbounded. We further define a map $h : \mathbb{N} \longrightarrow \mathbb{N}$ by Recursion as follows:

(i) $h(0) = g(0)$

(ii) $h(\sigma(m))$ is the minimum of $\{k \in \operatorname{ran} g \,:\, k > h(m)\}$ for $m \in \mathbb{N}$

We have $\operatorname{ran} h = \operatorname{ran} g$ by the Induction principle, and $x_{g(m)} \to x$. It follows that $x_{h(m)} \to x$. Moreover, h is strictly increasing by definition.

The converse is clear. □

The following result says that an increasing (decreasing) sequence that is bounded converges, provided certain conditions on the ordering are satisfied.

Lemma 6.10
Let $(X, <)$ be a totally ordered space where $<$ is an interval relation and has the least upper bound property, (x_n) a bounded sequence in X, and $A = \{x_n : n \in \mathbb{N}\}$. The following statements hold:

(i) If (x_n) is increasing, then $x_n \to \sup A$ with respect to the interval topology.

(ii) If (x_n) is decreasing, then $x_n \to \inf A$ with respect to the interval topology.

Proof. In order to show (i), assume that (x_n) is increasing. Since A has an upper bound, it has a supremum by assumption. This supremum is unique by Lemma and Definition 2.47. We define $x = \sup A$.
By Lemmas 5.99 and 5.88 the system

$$\mathcal{A} = \{\,]y, z[\; : \, y, z \in X, \; y < x < z\}$$

is a neighborhood base of x. Now let $y, z \in X$ with $y < x < z$. Assume there is no $m \in \mathbb{N}$ such that $x_m \in \;]y, z[$. Then y is an upper bound of A, which is a contradiction. Hence there is $n \in \mathbb{N}$ such that $x_m \in \;]y, z[$ for every $m \in \mathbb{N}$ with $m \geq n$ since (x_n) is increasing.
To see (ii), notice that A has a greatest lower bound if it has a lower bound, by Theorem 2.49. The remainder of the proof is similar to that of (i). □

Remark 6.11
Notice that $(\mathbb{R}, <)$ satisfies the conditions of Lemma 6.10 by Lemmas 5.102 and 4.39. ■

Lemma 6.12
Let (x_n) be a sequence in $\mathbb{R}_+\setminus\{0\}$. If (x_n) is increasing and unbounded, then $x_n^{-1} \to 0$ with respect to the standard topology on $\mathbb{R}$.

Proof. Let $y \in \mathbb{R}_+\setminus\{0\}$. Under the stated conditions, there is $m \in \mathbb{N}$ such that $x_n > y^{-1}$ for every $n \in \mathbb{N}$ with $n > m$. Hence $x_n^{-1} < y$ for $n \in \mathbb{N}$, $n > m$ by Corollary 4.27 and Remark 5.102. □

We may compare the convergence properties of a sequence with respect to two different topologies on the same set. To this end we introduce the following notions.

Definition 6.13
Let X be a set and $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then $\mathcal{T}_1$ is called **sequentially stronger than** $\mathcal{T}_2$ if $x_n \to x$ with respect to $\mathcal{T}_1$ implies $x_n \to x$ with respect to $\mathcal{T}_2$ for every sequence (x_n) in X and every $x \in X$. $\mathcal{T}_1$ and $\mathcal{T}_2$ are called **sequentially equivalent** if $\mathcal{T}_1$ is sequentially stronger than $\mathcal{T}_2$ and vice versa. ■

The following is an intuitive result that relates the comparison of two topologies on the same set to the comparison of sequence convergence.

Lemma 6.14
Let X be a set and $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. If $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, then it is also sequentially stronger.

Proof. Assume that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ and let (x_n) be a sequence in X, $x \in X$, and $x_n \to x$ with respect to $\mathcal{T}_1$. For $i \in \{1, 2\}$ let $\mathcal{N}_i$ be the neighborhood system of $(X, \mathcal{T}_i)$. We have $\mathcal{N}_2\{x\} \subset \mathcal{N}_1\{x\}$ by Lemma 5.97. Thus $x_n \to x$ with respect to $\mathcal{T}_2$. □

The condition under which the converse is true is proven in Section 6.5 because a result about the closure of sets is required that is not derived before.

Lemma and Definition 6.15
Let X and Y be two sets, $f : X \longrightarrow Y$ a map, $A \subset X$, and (x_n) a sequence in A. Then $(f(x_n) : n \in \mathbb{N})$ is a sequence in Y, also denoted by $(f(x_n))$ or $f(x_n)$.

Proof. This is clear. □

6.2 Nets

In this Section the concept of sequences is generalized by the introduction of nets.

Definition 6.16
Given a set X and a directed space $(D, \leq)$, a function $x : D \longrightarrow X$ is called a **net in** X. It is denoted by $(x_n : n \in D)$, or, if it is known from the context that x is a net with some domain D, by the short notations (x_n) or x_n. The value $x(n)$ of x at a point $n \in D$ is also denoted by x_n. ■

Note that $(\mathbb{N}, \leq)$ is a directed space. Therefore, given a set X, the nets in X that are of the form $(x_n : n \in \mathbb{N})$, i.e. those where the directed set is $\mathbb{N}$, are precisely the sequences in X. In this case the definitions of the notations $(x_n : n \in \mathbb{N})$, (x_n), and x_n for nets in Definition 6.16 agree with the respective definitions for sequences in Definition 6.1. Many of the illustrative properties of sequences generalize to similar properties of nets.

Definition 6.17
Let X be a set, $(x_n : n \in D)$ a net in X, and $\varphi(x)$ a formula. We say that $\varphi(x_n)$ is true **eventually**, or short $\varphi(x_n)$ **eventually**, if there is $N \in \mathbb{N}$ such that $\varphi(x_n)$ is true for every $n \in \mathbb{N}$ with $N \leq n$. We say that $\varphi(x_n)$ is true **frequently**, or short $\varphi(x_n)$ **frequently**, if, for every $N \in \mathbb{N}$, there is $n \in \mathbb{N}$ with $N \leq n$ such that $\varphi(x_n)$ is true. ■

Notice that in the case $D = \mathbb{N}$ Definition 6.17 agrees with Definition 6.2 where the same notions are defined for sequences. Regarding the usage of formula variables in Definition 6.17 the same remarks apply as with respect to Definition 6.2.

Definition 6.18
Given a topological space $(X, \mathcal{T})$, a subset $A \subset X$, and a net $(x_n : n \in D)$ in A, a point $x \in X$ is called **limit point of** (x_n) if $x_n \in U$ eventually for every $U \in \mathcal{N}\{x\}$. In this case we say that (x_n) **converges to** x, or write $x_n \to x$. The set of all limit points of (x_n) is denoted by $\lim_n x_n$. If (x_n) has a limit point, it is called **convergent**. If (x_n) has a unique limit point, say x, we also write $\lim_n x_n = x$. Further, we say that a point $x \in X$ is an **adherence point of** (x_n) if $x_n \in U$ frequently for every $U \in \mathcal{N}\{x\}$. The set of all adherence points of (x_n) is denoted by $\operatorname{adh}_n x_n$. If (x_n) has a unique adherence point, say x, we also write $\operatorname{adh}_n x_n = x$. ■

Notice that every limit point of a net (x_n) is also an adherence point. A net can have more than one limit point. We remark that in the case $D = \mathbb{N}$ Definition 6.18 agrees with the Definition 6.3.

Example 6.19
Let $(X, \mathcal{T})$ be a topological space, $x \in X$, and $\leq$ the relation on $\mathcal{N}\{x\}$ that is defined by

$$U \leq V \quad \Longleftrightarrow \quad V \subset U$$

Then $(\mathcal{N}\{x\}, \leq)$ is a directed space (which justifies the notation $\leq$). We may choose a point $x_U \in X$ for each $U \in \mathcal{N}\{x\}$. Then $(x_U : U \in \mathcal{N}\{x\})$ is a net in X and $x_U \to x$.
Similarly, if $\mathcal{B}$ is a neighborhood base of x and the relation $\leq$ on $\mathcal{B}$ is defined as above for $U, V \in \mathcal{B}$, then $(\mathcal{B}, \leq)$ is a directed space. We may choose a point $x_B \in X$ for each $B \in \mathcal{B}$. Then $(x_B : B \in \mathcal{B})$ is a net in X and $x_B \to x$. ■

The analogue of Lemma 6.6 also holds for nets.

Lemma 6.20
Given a topological space $(X, \mathcal{T})$ and a net $(x_n : n \in D)$ in X, a point $x \in X$ is a limit point of (x_n) iff there is a neighborhood base $\mathcal{B}$ of x such that $x_n \in B$ eventually for every $B \in \mathcal{B}$.

Proof. Exercise. □

In the case of a pseudo-metric topology we have the following characterization of convergence.

Lemma 6.21
Let (X, d) be a pseudo-metric space, $(x_n : n \in D)$ a net in X, and $x \in X$. (x_n) converges to x with respect to $\tau(d)$ iff the net $(d(x_n, x) : n \in D)$ converges to 0 with respect to the standard topology on $\mathbb{R}_+$.

Proof. Exercise. □

Remark 6.22
Let (x_n) be a net in $\mathbb{R}$ and $x \in \mathbb{R}$. Then we have:

$$x_n \to x \quad \Longleftrightarrow \quad |x_n - x| \to 0$$

where the limit on the left-hand side is with respect to the standard topology on $\mathbb{R}$ and the limit on the right-hand side is with respect to the standard topology on $\mathbb{R}_+$. ■

We now introduce subnets, which are the analogue of subsequences as defined in Definition 6.8.

Definition 6.23
Given two sets X and Y, and a net $(x_n : n \in D)$ in X, a net $(y_m : m \in E)$ in Y is called **subnet of** (x_n) if there is a map $f : E \longrightarrow D$ such that

(i) $\forall m \in E \quad y_m = x_{f(m)}$

(ii) $\forall n \in D \quad \exists m \in E \quad \forall p \in E \quad p \geq m \implies f(p) \geq n$

We also use the short notation $(y_m) \subset (x_n)$. ■

Lemma 6.24
Given a topological space $(X, \mathcal{T})$, a net (x_n) in X, and a subnet $(y_m) \subset (x_n)$, the following statements hold:

(i) $\lim_n x_n \subset \lim_m y_m$

(ii) $\operatorname{adh}_m y_m \subset \operatorname{adh}_n x_n$

Proof. Exercise. □

Lemma and Definition 2.85 can be used to derive the analogue of Theorem 6.9

for nets.

Theorem 6.25
Given a topological space $(X,\mathcal{T})$ and a net $(x_n : n \in D)$ in X, a point $x \in X$ is adherence point of (x_n) iff there is a subnet $(y_r : r \in E) \subset (x_n)$ such that $y_r \to x$.

Proof. Assume that $x \in \operatorname{adh}_n x_n$. Let the relation $\leq$ on $\mathcal{N}\{x\}$ be defined as in Example 6.19. We define

$$E = \big\{(n,U) \in D \times \mathcal{N}\{x\} \,:\, x_n \in U\big\}$$

and the relation $\leq$ on E as the restriction to E of the relation introduced in Lemma and Definition 2.85. Then $(E,\leq)$ is a directed space.

> [The relation $\leq$ is clearly transitive and reflexive. Let $(m,U),(n,V) \in E$, and $W = U \cap V$. Then we have $W \in \mathcal{N}\{x\}$, and there is $k \in \mathbb{N}$ such that $k \geq m,n$ and $x_k \in W$. It follows that $\leq$ is a directive relation on E.]

Further let $p : D \times \mathcal{N}\{x\} \longrightarrow D$ be the projection on the first component, and $(y_r : r \in E)$ the net in X such that $y = x \circ p$. Then (y_r) clearly is a subnet of (x_n). Furthermore, $y_r \to x$.
The converse is clear. □

The following Theorem states the existence of a diagonal net that has among its limit points the limit points of iterated limits.

Theorem 6.26 (Iterated limits)
Let $(X, \mathcal{T})$ be a topological space and $(D, \leq)$ a directed space. Further let $(E_m, \leq)$ be a directed space for every $m \in D$ and $F = \bigcup_{m \in D} \{m\} \times E_m$. Then there exists a net $(T_r : r \in R)$ in F such that for every map $S : F \longrightarrow X$ and every map $S' : D \longrightarrow X$ with $S'(m) \in \lim_n S(m, n)$ $(m \in D)$ we have $\lim_m S'(m) \subset \lim_r S \circ T(r)$. If $\mathcal{N}^{\text{closed}}$ is a neighborhood base, then equality holds instead of "$\subset$".

Proof. We define the product directed space $(R, \leq)$ where $R = D \times P$ and $P = \bigtimes_{m \in D} E_m$, the projections $p_m : P \longrightarrow E_m$ $(m \in D)$, and the net $T : R \longrightarrow F$, $T(m, e) = (m, p_m(e))$ for $m \in D$ and $e \in P$. Now let S and S' be two functions satisfying the stated conditions.
If $S'' \in \lim_m S'(m)$, and $U \in \mathcal{N}^{\text{open}}\{S''\}$, then there is $M \in D$ such that $S'(m) \in U$ for every $m \in D$ with $m \geq M$. We may choose $e \in P$ such that $S(m, n) \in U$ for every $m \in D$ with $m \geq M$ and every $n \in E_m$ with $n \geq p_m(e)$. Hence, $S \circ T(m, e') \in U$ for every $(m, e') \in R$ with $(m, e') \geq (M, e)$.
Conversely, assume that $\mathcal{N}^{\text{closed}}$ is a neighborhood base and let $S'' \in \lim_r S \circ T(r)$. Further let $U \in \mathcal{N}^{\text{closed}}\{S''\}$. There are $M \in D$, $e \in P$ such that $S(m, p_m(e')) \in U$ for every $(m, e') \in R$ with $(m, e') \geq (M, e)$. Then, for every $m \in D$ with $m \geq M$, we have $S(m, n) \in U$ for $n \in E_m$, $n \geq p_m(e)$. It follows that $\lim_n S(m, n) \subset U$ for every $m \in D$, $m \geq M$. Thus $S'' \in \lim_m S'(m)$. □

Lemma and Definition 6.27
Let X and Y be sets, $f : X \longrightarrow Y$ a map, $A \subset X$, and $(x_n : n \in D)$ a net in A. Then $(f(x_n) : n \in D)$ is a net in Y, and also denoted by $(f(x_n))$ or $f(x_n)$.

Proof. This is clear. □

The following construction is useful in the context of product spaces.

Proposition 6.28
Let (D_i, R_i) $(i \in I)$ be directed sets where I is an index set, (D, R) the product directed set, and $p_i : D \longrightarrow D_i$ $(i \in I)$ the projections. Further let $(X, \mathcal{T})$ be a topological space, $A \subset X$, $j \in I$, and $x : D_j \longrightarrow A$ a net in A. The net $y : D \longrightarrow A$, $y(r) = x\big(p_j(r)\big)$, is a subnet of (x_n). Moreover we have

$$\operatorname{ran} y = \operatorname{ran} x\,, \qquad \lim_n x_n = \lim_r y_r\,, \qquad \operatorname{adh}_n x_n = \operatorname{adh}_r y_r$$

Proof. Exercise. □

Definition 6.29
Let I be an index set. For every $i \in I$ let (D_i, R_i) be a directed set, $(X_i, \mathcal{T}_i)$ be a topological space, $A_i \subset X_i$, and $(x^i_n : n \in D_i)$ a net in A_i. Let (D, R) be the product directed set and $p_i : D \longrightarrow D_i$ $(i \in I)$ the projections. Further let $X = \bigtimes_{i\in I} X_i$ and $q_i : X \longrightarrow X_i$ $(i \in I)$ be the corresponding projections. Moreover let $A = \bigtimes_{i\in I} A_i$. The net $x : D \longrightarrow A$ defined by $q_i\big(x(r)\big) = x^i\big(p_i(r)\big)$ for every $i \in I$ and $r \in D$ is called **product net** and denoted by $\prod_{i\in I}(x^i_n)$. If $I = \sigma(n)\backslash m$ for some $m, n \in \mathbb{N}$ with $m < n$, then we also write $\prod_{k=m}^{n}(x^i_k)$ for the product net. If $I = \mathbb{N}\backslash m$ for some $m \in \mathbb{N}$, then we also write $\prod_{k=m}^{\infty}(x^i_k)$ for the product net. ■

Lemma 6.30
With definitions as in Definition 6.29 we have for every $i \in I$:

$$\lim_r q_i(x_r) = \lim_n x^i_n\,, \qquad \operatorname{adh}_r q_i(x_r) = \operatorname{adh}_n x^i_n$$

Proof. This follows by Propositon 6.28. □

6.3 Filters

We have introduced the notion of filter in Section 5.3. The concept of filter can be widely used as an alternative to nets when probing convergence and related properties. Although in many cases the usage of nets seems more illustrative, there are cases where filters are advantageous.

Definition 6.31
Given a topological space $(X, \mathcal{T})$ and a filter $\mathcal{F}$ on X, a point $x \in X$ is called **limit point of** $\mathcal{F}$ if $\mathcal{N}\{x\} \subset \mathcal{F}$. In this case we say that $\mathcal{F}$ **converges to** x, written $\mathcal{F} \to x$. If $\mathcal{F}$ has a limit point, it is called **convergent**. The set of all limit points of $\mathcal{F}$ is denoted by $\lim \mathcal{F}$. Further, a point $x \in X$ is called an **adherence point of** $\mathcal{F}$ if $U \cap F \neq \emptyset$ for every $U \in \mathcal{N}\{x\}$ and every $F \in \mathcal{F}$. The set of all adherence points of $\mathcal{F}$ is denoted by $\operatorname{adh} \mathcal{F}$. Let $\mathcal{B} \subset \mathcal{F}$ be a filter base for $\mathcal{F}$. A point $x \in X$ is called **limit point of** $\mathcal{B}$ if it is a limit point of $\mathcal{F}$. In this case we say that $\mathcal{B}$ **converges to** x, written $\mathcal{B} \to x$. A point $x \in X$ is called **adherence point of** $\mathcal{B}$ if it is an adherence point of $\mathcal{F}$. The set of all limit (adherence) points of $\mathcal{B}$ is denoted by $\lim \mathcal{B}$ ($\operatorname{adh} \mathcal{B}$). ■

Example 6.32
Given a topological space $(X, \mathcal{T})$ and a point $x \in X$, we obviously have $\mathcal{N}\{x\} \to x$. ■

Example 6.33
The filter in Example 5.63 has no adherence points. ■

We now establish a connection between sequences and filters, similarly to that between sequences and nets in Section 6.2. The agreement of the respective limit and adherence points is shown.

Lemma and Definition 6.34
Given a set X and a sequence (x_n) in X, the system

$$\mathcal{F} = \{F \subset X \ : \ x_n \in F \text{ eventually}\}$$

is a filter. We say that the sequence **generates** $\mathcal{F}$. The system $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ where $B_n = \{x_k : k \geq n\}$ $(n \in \mathbb{N})$ is a filter base for $\mathcal{F}$. We also say that the sequence **generates** $\mathcal{B}$.

Proof. $\mathcal{B}$ is a filter base by Lemma and Definition 5.55. Clearly, $\mathcal{F} = \Phi(\mathcal{B})$. □

Lemma 6.35
Let $(X, \mathcal{T})$ be a topological space, (x_n) a sequence in X, and $\mathcal{F}$ the filter generated by (x_n). We have $\lim_n x_n = \lim \mathcal{F}$ and $\operatorname{adh}_n x_n = \operatorname{adh} \mathcal{F}$.

Proof. Exercise. □

For a given set X, Lemma and Definition 6.34 introduces an injective map from the system of all sequences in X to the system of all filters on X, which is clearly not bijective in general.
Next, two types of connections between nets and filters on a set X are established and the relationship between their limit and adherence points is investigated. In the first case a net, the so-called associated net, is defined for a given filter. However, the limit points and the adherence points of the filter and the associated net are generally not the same.

Lemma and Definition 6.36

Let X be a set and denote by $\leq$ the relation $\supset$ on $\mathcal{P}(X)$. Further let $\mathcal{F}$ be a filter on X and $\mathcal{B}$ a filter base for $\mathcal{F}$. Then both $(\mathcal{B}, \leq)$ and $(\mathcal{F}, \leq)$ are directed spaces. We may choose a point $x_F \in F$ for each $F \in \mathcal{F}$. Then $(x_F : F \in \mathcal{F})$ is a net in X. Similarly, we may choose a point $x_B \in B$ for each $B \in \mathcal{B}$. Then $(x_B : B \in \mathcal{B})$ is a net in X. If a net in X can be constructed in this way, it is called **associated with** $\mathcal{F}$ or $\mathcal{B}$, respectively.

Proof. This is obvious. □

Lemma 6.37

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{F}$ a filter on X, $\mathcal{B}$ a filter base for $\mathcal{F}$, and $(x_F : F \in \mathcal{F})$ and $(x_B : B \in \mathcal{B})$ nets associated with $\mathcal{F}$ and $\mathcal{B}$, respectively. Then the following two statements hold:

(i) $\lim \mathcal{F} \subset \lim_F x_F$

(ii) $\lim \mathcal{F} \subset \lim_B x_B$

(iii) $\operatorname{adh}_F x_F \subset \operatorname{adh} \mathcal{F}$

(iv) $\operatorname{adh}_B x_B \subset \operatorname{adh} \mathcal{F}$

Proof. To see (i), let $x \in \lim \mathcal{F}$ and $U \in \mathcal{N}\{x\}$. Since $U \in \mathcal{F}$, it follows that $x_V \in U$ for $V \in \mathcal{F}$, $V \geq U$.

To show (ii), let $x \in \lim \mathcal{F}$ and $U \in \mathcal{N}\{x\}$. Since $U \in \mathcal{F}$, there is $B \in \mathcal{B}$ such that $B \subset U$. It follows that $x_A \in U$ for $A \in \mathcal{B}$, $A \geq B$.

In order to prove (iii), let $x \in \operatorname{adh}_F x_F$, $F \in \mathcal{F}$, and $U \in \mathcal{N}\{x\}$. Then there is x_G with $G \geq F$ such that $x_G \in U$. Since we also have $x_G \in F$, it follows that $F \cap U \neq \emptyset$.

To see (iv), let $x \in \operatorname{adh}_B x_B$, $F \in \mathcal{F}$, and $U \in \mathcal{N}\{x\}$. We may choose $B \in \mathcal{B}$ such that $B \subset F$. Then there is x_A with $A \geq B$ such that $x_A \in U$. Since we also have $x_A \in F$, it follows that $F \cap U \neq \emptyset$. □

We now show how a filter on X can be defined for a given net in X, and a net in X can be defined for a given filter on X, such that when starting with any filter and performing both steps the original filter is obtained. Note that generally there does not exist any bijection between all nets in X and all filters on X. In fact, there exists no set that contains every directed space, and therefore there is no set that contains every net in X. However, limit points and adherence points agree between corresponding nets and filters in this approach, so a true equivalence of the two concepts is demonstrated.

Lemma and Definition 6.38
Given a set X and a net $(x_n : n \in D)$ in X, the system

$$\mathcal{F} = \big\{ F \subset X \,:\, x_n \in F \text{ eventually} \big\}$$

is a filter. We say that (x_n) **generates** $\mathcal{F}$. The system $\mathcal{B} = \{B_n : n \in D\}$ where $B_n = \{x_k : k \geq n\}$ $(n \in D)$ is a filter base for $\mathcal{F}$. We also say that (x_n) **generates** $\mathcal{B}$.

Proof. $\mathcal{B}$ is a filter base by Lemma and Definition 5.55. Clearly, we have $\mathcal{F} = \Phi(\mathcal{B})$. □

Notice that Lemma and Definition 6.38 does not refer to any topology on X.

Lemma 6.39
Let $(X, \mathcal{T})$ be a topological space, (x_n) a net in X, $\mathcal{F}$ the filter generated by (x_n), and $x \in X$. We have $\lim_n x_n = \lim \mathcal{F}$ and $\operatorname{adh}_n x_n = \operatorname{adh} \mathcal{F}$.

Proof. Exercise. □

Lemma and Definition 6.40
Let X be a set, $\mathcal{F}$ a filter on X, $\mathcal{B}$ a filter base for $\mathcal{F}$, and

$$D = \{(x, B) : x \in B \in \mathcal{B}\}$$

Let the relation $\leq$ on D be defined as follows:

$$(x, B) \leq (y, C) \quad \Longleftrightarrow \quad C \subset B$$

Then $(D, \leq)$ is a directed space (which justifies the notation). The net $(x_n : n \in D)$ in X where $x_{(x,B)} = x$ for every $(x, B) \in D$ is called **generated by $\mathcal{B}$.**
The filter generated by (x_n) is $\mathcal{F}$.

Proof. $(D, \leq)$ clearly is a directed space.
Let $\mathcal{G}$ be the filter generated by (x_n).
To see that $\mathcal{G} \subset \mathcal{F}$, let $G \in \mathcal{G}$. Then there is $n \in D$ such that $x_k \in G$ for $k \in D$, $k \geq n$. Moreover there is $B \in \mathcal{B}$ such that $n = (x_n, B)$. Then we have $x_k \in B$ for $k \in D$, $k \geq n$. Now let $x \in B$. Then $(x, B) \geq (x_n, B)$, and therefore $x = x_{(x,B)} \in G$. Thus we have $B \subset G$. It follows that $G \in \mathcal{F}$.
Conversely, let $B \in \mathcal{B}$. We may choose $x \in B$ and define $n = (x, B)$. It follows that $x_k \in B$ for $k \in D$, $k \geq n$. Thus we obtain $B \in \mathcal{G}$. □

Notice that also Lemma and Definition 6.40 does not refer to any topology on X. The following is the analogue to Theorems 6.9 and 6.25 where similar results for sequences and nets are proven, respectively.

Theorem 6.41
Let $(X, \mathcal{T})$ be a topological space and $\mathcal{F}$ a filter on X. A point $x \in X$ is an adherence point of $\mathcal{F}$ iff there is a filter $\mathcal{G}$ finer than $\mathcal{F}$ such that $\mathcal{G} \to x$.

Proof. First let $x \in \operatorname{adh} \mathcal{F}$. Then the system

$$\mathcal{G} = \{F \cap U \,:\, F \in \mathcal{F},\, U \in \mathcal{N}\{x\}\}$$

is a filter. Moreover, we have $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{G} \to x$.
Conversely, if there exists a filter $\mathcal{G}$ finer than $\mathcal{F}$ with $\mathcal{G} \to x$, then $\mathcal{N}\{x\} \subset \mathcal{G}$. It follows that $F \cap U \neq \emptyset$ for every $F \in \mathcal{F}$ and $U \in \mathcal{N}\{x\}$. □

We now briefly analyse how filter bases behave under mappings und thereby introduce the notions of image filter and inverse image filter.

Lemma and Definition 6.42
Let X and Y be two sets, $\mathcal{B}$ a filter base on X, and $f : X \longrightarrow Y$ a map. Then $f\,[\![\mathcal{B}]\!]$ is a filter base on Y. The generated filter $\Phi\left(f\,[\![\mathcal{B}]\!]\right)$ is called **image filter of $\mathcal{B}$ under f**.

Proof. Exercise. □

Notice that, even if $\mathcal{B}$ in Lemma and Definition 6.42 is a filter on X, the image $f\,[\![\mathcal{B}]\!]$ need not be a filter on Y.

Lemma 6.43
With definitions as in Lemma and Definition 6.42 we have $\Phi\left(f\,[\![\mathcal{B}]\!]\right) = \Phi\left(f\,[\![\Phi(\mathcal{B})]\!]\right)$, that is the image filter of a filter base and of its generated filter are the same.

Proof. Exercise. □

Lemma and Definition 6.44
Let X and Y be two sets, $\mathcal{B}$ a filter base on Y, and $f : X \longrightarrow Y$ a surjective map. Then $f^{-1}\,[\![\mathcal{B}]\!]$ is a filter base for a filter on X. The generated filter $\Phi\left(f^{-1}\,[\![\mathcal{B}]\!]\right)$ is called **inverse image filter of $\mathcal{B}$ under f**.

Proof. Exercise. □

Again, $f^{-1}[\![\mathcal{B}]\!]$ need not be a filter on X, even if $\mathcal{B}$ is a filter on Y.

6.4 Continuous functions

The topic of this Section are continuous functions. We introduce three types of continuity and then show how they are interrelated. The first one does not refer to any topology or pseudo-metric. It is based on two filters, one on the domain of the function and the other on its range. Moreover, it is a local definition, i.e. it refers to a single point of the domain. The second definition is based on two topologies, one on the domain and the other on the range. There we introduce both continuity in a point and continuity of the whole function. The third type of continuity is introduced in the context of pseudo-metric spaces and a priori does not refer to any topology. Also in this case a local and global form of continuity is defined. It is then shown that continuity with respect to pseudo-metrics is equivalent to continuity with respect to the generated topologies. Finally we analyse the special case of interval topologies.

Lemma and Definition 6.45
Let X, Y be two sets, $f : X \longrightarrow Y$ a function, $x \in X$, $y = f(x)$, $\mathcal{F}_x$ a filter on X such that x is a cluster point of $\mathcal{F}_x$, and $\mathcal{F}_y$ a filter on Y such that y is a cluster point of $\mathcal{F}_y$. f is called $\mathcal{F}_x$-$\mathcal{F}_y$-**continuous in** x if $\mathcal{F}_y \subset_\Phi f[\![\mathcal{F}_x]\!]$.
Further, let $\mathcal{B}_x$ and $\mathcal{B}_y$ be filter bases for $\mathcal{F}_x$ and $\mathcal{F}_y$, respectively. Then f is $\mathcal{F}_x$-$\mathcal{F}_y$-continuous in x iff $\mathcal{B}_y \subset_\Phi f[\![\mathcal{B}_x]\!]$.

Proof. Exercise. □

Definition 6.46

Given two topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, a map $f : X \longrightarrow Y$, and a point $x \in X$, f is called $\mathcal{T}_X$**-**$\mathcal{T}_Y$**-continuous in** x, or short **continuous in** x, if $f^{-1}[\![\, \mathcal{T}_Y(f(x)) \,]\!] \subset \mathcal{T}_X$. ■

The following Theorem shows how the two types of continuity are related and provides various characterizations of continuity of a function in a point.

Theorem 6.47

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a map $f : X \longrightarrow Y$, a subbase $\mathcal{S}_Y$ for $\mathcal{T}_Y$, a point $x \in X$, and a neighborhood base $\mathcal{E}$ of x, the following statements are equivalent:

(i) f is continuous in x.

(ii) $f^{-1}[\![\, \mathcal{S}_Y(f(x)) \,]\!] \subset \mathcal{T}_X$

(iii) $\mathcal{S}_Y(f(x)) \subset_\Phi f[\![\mathcal{E}\,]\!]$

(iv) $f^{-1}[\![\mathcal{N}\{f(x)\}]\!] \subset \mathcal{N}\{x\}$

(v) $\mathcal{N}\{f(x)\} \subset_\Phi f[\![\mathcal{N}\{x\}]\!]$, i.e. f is $\mathcal{N}\{x\}$-$\mathcal{N}\{f(x)\}$-continuous in x.

(vi) For every filter base $\mathcal{B}$ on X, $\mathcal{B} \to x$ implies $f[\![\mathcal{B}\,]\!] \to f(x)$.

(vii) For every net (x_n) in X, $x_n \to x$ implies $f(x_n) \to f(x)$.

Proof. We show the equivalence of (i) to (iv), and then the equivalence of (iv) to (vii).

The implication from (i) to (ii) is clear.

To show that(ii) implies (iii), let $S \in \mathcal{S}_Y(f(x))$. Then we have $f^{-1}[S] \in \mathcal{T}_X$ by assumption. We may choose $B \in \mathcal{E}$ such that $B \subset f^{-1}[S]$. It follows that

$$f[B] \subset f\left[f^{-1}[S]\right] \subset S$$

To see that (iii) implies (iv), let $U \in \mathcal{N}\{f(x)\}$. There are $S_i \in \mathcal{S}_Y$ $(i \in I)$, where I is a finite index set, such that $f(x) \in S_i$ for every $i \in I$ and $S \subset U$ where $S = \bigcap_{i \in I} S_i$. For every $i \in I$, we may choose $B_i \in \mathcal{E}$ such that $f[B_i] \subset S_i$ by assumption. We define $B = \bigcap_{i \in I} B_i$. It follows that $x \in B \subset f^{-1}[U]$.
To show that (iv) implies (i), let $U \in \mathcal{T}_Y$ with $f(x) \in U$. Further let $z \in f^{-1}[U]$. By Lemma 5.77 (i) we have $U \in \mathcal{N}\{f(z)\}$, and therefore $f^{-1}[U] \in \mathcal{N}\{z\}$ by assumption. Thus $f^{-1}[U] \in \mathcal{T}_X$ by Lemma 5.77 (i).
To show that (iv) implies (vii), let $(x_n : n \in D)$ be a net in X with $x_n \to x$, and let $U \in \mathcal{N}\{f(x)\}$. By assumption we have $f^{-1}[U] \in \mathcal{N}\{x\}$. There is $n \in D$ such that $x_k \in f^{-1}[U]$ for $k \in D$, $k \geq n$. It follows that $f(x_k) \in U$ for $k \in D$, $k \geq n$.
To prove that (vii) implies (vi), let $\mathcal{B}$ be a filter base on X with $\mathcal{B} \to x$. Further let $\mathcal{F} = \Phi(\mathcal{B})$, $(x_n : n \in D)$ the net generated by $\mathcal{F}$, and $U \in \mathcal{N}\{f(x)\}$. It follows that $x_n \to x$, and thus $f(x_n) \to f(x)$ by assumption. Let $(y_m : m \in E)$ be the net generated by $f[\![\mathcal{F}]\!]$. Then we have $\lim_n f(x_n) \subset \lim_m y_m$.

[Let $r \in \lim_n f(x_n)$ and $U \in \mathcal{N}\{r\}$. We may choose $(z, F) = n \in D$ such that $f(x_k) \in U$ for $k \in D$, $k \geq n$. Let $e = (f(z), f[F])$. We clearly have $e \in E$. Now let $(v, V) = g \in E$ with $g \geq e$. We define $C = F \cap f^{-1}[V]$. There is $A \in \mathcal{F}$ such that $f[A] = V$. Since $F \cap A \in \mathcal{F}$ and $A \subset f^{-1}[V]$, we have $C \in \mathcal{F}$. Further we have $f[C] = V$ (exercise). Hence there is $c \in C$ with $f(c) = v$. Thus we have $g = (f(c), f[C])$. Since $(c, C) \geq (z, F)$, we have $y_g = f(c) = f\left(x_{(c,C)}\right) \in U$.]

It follows that $f[\![\mathcal{F}]\!] \to f(x)$, and thus $f[\![\mathcal{B}]\!] \to f(x)$ by Lemma 6.43.
To prove that (vi) implies (v), note that $\mathcal{N}\{x\}$ is a filter base on X with $\mathcal{N}\{x\} \to x$. Thus we have $f[\![\mathcal{N}\{x\}]\!] \to f(x)$ by assumption. Let $U \in \mathcal{N}\{f(x)\}$. It follows that there is $V \in \mathcal{N}\{x\}$ such that $f[V] \subset U$.

To see that (v) implies (iv), let $U \in \mathcal{N}\{f(x)\}$. By assumption there is $V \in \mathcal{N}\{x\}$ such that $f[V] \subset U$. It follows that $V \subset f^{-1}[U]$, and thus $f^{-1}[U] \in \mathcal{N}\{x\}$. □

Definition 6.48

Given two topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and a map $f : X \longrightarrow Y$, f is called $\mathcal{T}_X$**-**$\mathcal{T}_Y$**-continuous**, or short **continuous**, if f is continuous in x for every $x \in X$. If f is bijective and if both f and f^{-1} are continuous, then f is called a $\mathcal{T}_X$**-**$\mathcal{T}_Y$**-homeomorphism**, or short **homeomorphism**. ■

Theorem 6.49

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces, $\mathcal{C}_X$ and $\mathcal{C}_Y$ the systems of all closed subsets of X and Y, respectively, $\mathcal{S}_X$ a subbase for $\mathcal{T}_X$, and $f : X \longrightarrow Y$ a map, the following statements are equivalent:

(i) f is continuous.

(ii) $f^{-1}[\![\mathcal{S}_Y]\!] \subset \mathcal{T}_X$

(iii) $\forall x \in X \quad f^{-1}[\![\mathcal{N}\{f(x)\}]\!] \subset \mathcal{N}\{x\}$

(iv) $\forall x \in X \quad \mathcal{N}\{f(x)\} \subset_\Phi f[\![\mathcal{N}\{x\}]\!]$, i.e. f is $\mathcal{N}\{x\}$-$\mathcal{N}\{f(x)\}$-continuous in x for every $x \in X$.

(v) For every $x \in X$ and every filter base $\mathcal{B}$ on X, $\mathcal{B} \to x$ implies $f[\![\mathcal{B}]\!] \to f(x)$.

(vi) For every $x \in X$ and every net (x_n) in X, $x_n \to x$ implies $f(x_n) \to f(x)$.

(vii) $f^{-1}[\![\mathcal{T}_Y]\!] \subset \mathcal{T}_X$

(viii) $f^{-1}[\![\mathcal{C}_Y]\!] \subset \mathcal{C}_X$

Proof. The equivalence of (i) to (vi) follows by Theorem 6.47.

The equivalence of (i) and (vii) is obvious.

Finally, the equivalence of (vii) and (viii) follows by complementation. □

Further equivalent statements are proven in Theorem 6.88 below.

Remark 6.50
Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ iff id_X is $\mathcal{T}_1$-$\mathcal{T}_2$-continuous. ■

In the case of a first countable domain space, continuity may be characterized through the convergence of sequences as follows.

Lemma 6.51
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces where $\mathcal{T}_X$ is first countable, $x \in X$, and $f : X \longrightarrow Y$ a map. Then the following statements are equivalent:

(i) f is continuous in x.

(ii) For every sequence (x_n) in X, $x_n \to x$ implies $f(x_n) \to f(x)$.

Proof. (i) clearly implies (ii).
We prove that (ii) implies (i). By Lemma 5.96 we may choose a neighborhood base $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ of x such that all B_n $(n \in \mathbb{N})$ are open and $B_n \subset B_m$ for every $m, n \in \mathbb{N}$ with $m < n$. Assume that (ii) is true and that f is not continuous in x. Then there is $U \in \mathcal{N}\{f(x)\}$ such that $f^{-1}[U] \notin \mathcal{N}\{x\}$ by Theorem 6.47. We may choose a sequence (x_n) such that $x_n \in B_n \setminus f^{-1}[U]$ for every $n \in \mathbb{N}$. It follows that $x_n \to x$, and thus $f(x_n) \to f(x)$ by assumption. Therefore there is $m \in \mathbb{N}$ such that $f(x_n) \in U$ for $n \in \mathbb{N}$, $n \geq m$. Hence $x_n \in f^{-1}[U]$ for $n \in \mathbb{N}$ with $n \geq m$, which is a contradiction. □

Definition 6.52

Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are called **homeomorphic** if there exists a homeomorphism $f : X \longrightarrow Y$. ■

Lemma 6.53

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be homeomorphic topological spaces and let $f : X \longrightarrow Y$ be a homeomorphism. Then $f\,[\![\mathcal{T}_X]\!] = \mathcal{T}_Y$ and $f^{-1}\,[\![\mathcal{T}_Y]\!] = \mathcal{T}_X$.

Proof. We have $f^{-1}\,[\![\mathcal{T}_Y]\!] \subset \mathcal{T}_X$ by the continuity of f, and $f\,[\![\mathcal{T}_X]\!] \subset \mathcal{T}_Y$ by the continuity of f^{-1}. We define the functions

$$F : \mathcal{T}_X \longrightarrow \mathcal{T}_Y\,, \quad F(A) = f\,[A]\,;$$

$$G : \mathcal{T}_Y \longrightarrow \mathcal{T}_X\,, \quad G(B) = f^{-1}\,[B]$$

It follows that

$$G \circ F(A) = f^{-1}\,[f\,[A]] = A\,, \qquad F \circ G(B) = f\left[f^{-1}\,[B]\right] = B$$

for every $A \in \mathcal{T}_X$ and $B \in \mathcal{T}_Y$. Therefore F and G are bijective. Thus

$$f\,[\![\mathcal{T}_X]\!] = F\,[\mathcal{T}_X] = \mathcal{T}_Y\,, \qquad f^{-1}\,[\![\mathcal{T}_Y]\!] = G\,[\mathcal{T}_Y] = \mathcal{T}_X$$

□

Example 6.54

Let a, b be two sets, and $X = \{a, b\}$. Then the systems

$$\mathcal{T}_a = \big\{\emptyset, X, \{a\}\big\}\,, \qquad \mathcal{T}_b = \big\{\emptyset, X, \{b\}\big\}$$

are topologies on X. The topological spaces $(X, \mathcal{T}_a)$ and $(X, \mathcal{T}_b)$ are homeomorphic, and the function

$$f : X \longrightarrow X\,, \quad f(a) = b, \quad f(b) = a$$

is a $\mathcal{T}_a$-$\mathcal{T}_b$-homeomorphism. If $a \neq b$, then $\mathcal{T}_a \neq \mathcal{T}_b$. ■

Lemma 6.53 says that two homeomorphic topological spaces are essentially the same, that is they have all properties in common that are related to their topologies. For example if a topological space is first or second countable, also its homeomorphic counterpart is first or second countable, respectively. In some cases a subspace of a topological space whose properties are known and the topological space to be analysed are homeomorphic. Then the space in question automatically has those properties that are inherited by the subspace. The following Definition is suitable for such cases.

Definition 6.55
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and $f : X \longrightarrow Y$ a map. Then f is called **embedding of X in Y** if the map $g : X \longrightarrow f[X]$, $g(x) = f(x)$, is a homeomorphism. ■

We now introduce the third type of continuity, both in a point and globally, viz. for pseudo-metric spaces.

Definition 6.56
Given two pseudo-metric spaces (X, d_X), (Y, d_Y), a map $f : X \longrightarrow Y$, and a point $x \in X$, f is called **d_X-d_Y-continuous in x**, or short **continuous in x**, if

$$\forall \varepsilon \in \,]0, \infty[\quad \exists \, \delta \in \,]0, \infty[\quad \forall y \in X \quad d_X(x, y) < \delta \implies d_Y\big(f(x), f(y)\big) < \varepsilon$$

If f is continuous in x for every $x \in X$, then f is called **d_X-d_Y-continuous**, or short **continuous**. ■

The following Lemma states that the definitions of continuity for pseudo-metric spaces and for topological spaces are in agreement with the generation of a topology by a pseudo-metric.

Lemma 6.57
Let (X, d_X) and (Y, d_Y) be pseudo-metric spaces, respectively, $f : X \longrightarrow Y$ a map, and $x \in X$. f is d_X-d_Y-continuous in x iff f is $\tau(d_X)$-$\tau(d_Y)$-continuous in x. Further, f is d_X-d_Y-continuous iff f is $\tau(d_X)$-$\tau(d_Y)$-continuous.

Proof. To prove the first claim, it is enough to show that f is d_X-d_Y-continuous in x iff f is $\mathcal{N}\{x\}$-$\mathcal{N}\{f(x)\}$-continuous in x by Theorem 6.47.
First assume that f is d_X-d_Y-continuous in x, and let $U \in \mathcal{N}\{f(x)\}$. By Lemma and Definition 5.119 there is $r \in \,]0, \infty[$ such that $B \subset U$ where B is the open sphere about $f(x)$ with d_Y-radius r. It follows that there is $s \in \,]0, \infty[$ such that $f[A] \subset B$ where A is the open sphere about x with d_X-radius s. Since $A \in \mathcal{N}\{x\}$, this shows that f is $\mathcal{N}\{x\}$-$\mathcal{N}\{f(x)\}$-continuous.
Now assume that f is $\mathcal{N}\{x\}$-$\mathcal{N}\{f(x)\}$-continuous. Let $r \in \,]0, \infty[$ and B the open sphere about $f(x)$ with d_Y-radius r. Then we have $B \in \mathcal{N}\{f(x)\}$. There is $C \in \mathcal{N}\{x\}$ such that $f[C] \subset B$ by assumption. By Lemma and Definition 5.119 there is $s \in \,]0, \infty[$ such that $A \subset C$ where A is the open sphere about x with d_Y-radius r. It follows that f is d_X-d_Y-continuous in x.
The second claim follows by the first one. □

Both with respect to filters and with respect to topologies, the composition of two continuous functions is continuous.

Lemma 6.58
Let $(X_i, \mathcal{T}_i)$ $(i \in \{1, 2, 3\})$ be topological spaces, $f : X_1 \longrightarrow X_2$ and $g : X_2 \longrightarrow X_3$ maps, $x_1 \in X_1$ a point, $x_2 = f(x_1)$, and $x_3 = g(x_2)$. For every $i \in \{1, 2, 3\}$ let $\mathcal{F}_i$ be a filter on X_i such that x_i is a cluster point of $\mathcal{F}_i$. If f is $\mathcal{F}_1$-$\mathcal{F}_2$-continuous in x_1 and g is $\mathcal{F}_2$-$\mathcal{F}_3$-continuous in x_2, then $g \circ f$ is $\mathcal{F}_1$-$\mathcal{F}_3$-continuous in x_1.

Proof. Exercise. □

Lemma 6.59
Let $(X_i, \mathcal{T}_i)$ $(i \in \{1, 2, 3\})$ be topological spaces, $f : X_1 \longrightarrow X_2$ and $g : X_2 \longrightarrow X_3$ maps, and $x \in X_1$. If f and g are continuous, then $g \circ f$ is continuous. If f is continuous in x, and g is continuous in $f(x)$, then $g \circ f$ is continuous in x.

Proof. To see the second claim, assume that f is $\mathcal{T}_1$-$\mathcal{T}_2$-continuous in x and g is $\mathcal{T}_2$-$\mathcal{T}_3$-continuous in $f(x)$. Let $A \in \mathcal{T}_3(g(f(x)))$. It follows that $g^{-1}[A] \in \mathcal{T}_2(f(x))$ by the continuity of g, and

$$(g \circ f)^{-1}[A] = f^{-1}\left[g^{-1}[A]\right] \in \mathcal{T}_1$$

Now the first claim clearly follows. □

We adapt the following convention.

Definition 6.60
Let $f \subset X \times Y$ be a function. If $X = \mathbb{R}$ or $Y = \mathbb{R}$, then continuity of f refers to the standard topology on $\mathbb{R}$ unless otherwise specified. ■

Definition 6.61
Given a pseudo-metric space (X, d), the function dist_d, or short dist, defined by

$$\mathrm{dist}_d : \big(\mathcal{P}(X)\backslash\{\emptyset\}\big) \times \big(\mathcal{P}(X)\backslash\{\emptyset\}\big) \longrightarrow \mathbb{R}$$

$$\mathrm{dist}_d(A, B) = \inf\big\{d(x, y) \,:\, x \in A,\, y \in B\big\}$$

is called **distance**. If one of the arguments is a singleton, we also write $\mathrm{dist}_d(A, x)$ and $\mathrm{dist}_d(x, B)$ instead of $\mathrm{dist}_d(A, \{x\})$ and $\mathrm{dist}_d(\{x\}, B)$, respectively, where $x \in X$. ■

Notice that the range of the distance is clearly a subset of $\mathbb{R}_+$.

Lemma 6.62
Given a pseudo-metric space (X, d) and a set $A \subset X$ where $A \neq \emptyset$, the function $f : X \longrightarrow \mathbb{R}$, $f(x) = \operatorname{dist}(A, x)$ is continuous.

Proof. Let $x \in X$ and $U \in \mathcal{N}\{f(x)\}$. We may choose $\varepsilon \in \mathbb{R}$ such that

$$]f(x) - \varepsilon,\ f(x) + \varepsilon[\ \subset U$$

Let $u, v \in X$. Then we have

$$d(x, v) \leq d(x, u) + d(u, v), \qquad d(u, v) \leq d(u, x) + d(x, v)$$

and hence

$$f(x) \leq d(x, u) + f(u), \qquad f(u) \leq d(x, u) + f(x)$$

by Lemmas 4.49 and 2.76. It follows that $|f(x) - f(u)| \leq d(x, u)$. Thus $f\,[B(x, \varepsilon)] \subset U$. $\square$

In the case of interval topologies, continuity can by characterized in the following way.

Remark 6.63

Let X and Y be two sets, $\mathcal{R}$ and $\mathcal{S}$ systems of pre-orderings on X and Y, respectively, $f : X \longrightarrow Y$ a function, and $x \in X$. The following statements are equivalent by Theorem 6.47 (iii):

(i) f is continuous in x.

(ii) For every $S \in \mathcal{S}$ and $y \in Y$ with $(y, f(x)) \in S$ there is a finite index set K and, for every $k \in K$, a pre-ordering $R_k \in \mathcal{R}$, a point $x_k \in X$, and an interval with either $I_k = \left]-\infty, x_k\right[_{R(k)}$ or $I_k = \left]x_k, \infty\right[_{R(k)}$, such that $x \in \bigcap_{k\in K} I_k$ and $f\left[\bigcap_{k\in K} I_k\right] \subset \left]y, \infty\right[_S$.

Moreover, for every $S \in \mathcal{S}$ and $y \in Y$ with $(f(x), y) \in S$ there is a finite index set K and, for every $k \in K$, a pre-ordering $R_k \in \mathcal{R}$, a point $x_k \in X$, and an interval with either $I_k = \left]-\infty, x_k\right[_{R(k)}$ or $I_k = \left]x_k, \infty\right[_{R(k)}$, such that $x \in \bigcap_{k\in K} I_k$ and $f\left[\bigcap_{k\in K} I_k\right] \subset \left]-\infty, y\right[_S$.

■

Finally, we introduce the notions of open and closed maps in this Section, which are, for example, relevant in the proof of Theorem 7.53 where a metric space is generated from a pseudo-metric space.

Definition 6.64

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces, $\mathcal{C}_X$ and $\mathcal{C}_Y$ the systems of all $\mathcal{T}_X$-closed and all $\mathcal{T}_Y$-closed sets, respectively, and $f : X \longrightarrow Y$ a map. If $f\left[\!\left[\mathcal{T}_X\right]\!\right] \subset \mathcal{T}_Y$, then f is called $\mathcal{T}_X$-$\mathcal{T}_Y$-**open**, or short **open**. If $f\left[\!\left[\mathcal{C}_X\right]\!\right] \subset \mathcal{C}_Y$, then f is called $\mathcal{T}_X$-$\mathcal{T}_Y$-**closed**, or short **closed**. ■

Lemma 6.65
Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be two topological spaces, $\mathcal{B}$ a base for $\mathcal{T}_X$, and $f : X \longrightarrow Y$ a map. Then the following statements are equivalent:

(i) f is open.

(ii) $f [\![\mathcal{B}]\!] \subset \mathcal{T}_Y$

Proof. Note that (ii) implies (i) by Lemma 2.63 (i). □

Lemma 6.66
With definitions as in Lemma 6.57, if f is an isometry, then f is continuous and open.

Proof. Assume that f is an isometry. Let $x \in X$, $y \in Y$, and $r \in \,]0, \infty[$. We have $f\left[B(x, r)\right] = B(f(x), r)$. Thus f is open by Lemma 6.65. Moreover, we have $f^{-1}\left[B(y, r)\right] = \bigcup \left\{B(z, r) : z \in f^{-1}\{y\}\right\}$. Hence f is continuous by Theorem 6.49 (ii). □

6.5 Closure, interior, derived set, boundary

We now investigate, for a given topological space $(X, \mathcal{T})$ and an arbitrary subset $A \subset X$, points whose neighborhood system has particular properties with respect to the set A. Specifically, we introduce the notions of interior points and their ensemble, called the interior of the set, accumulation points and their ensemble, the derived set, boundary points and their ensemble, called the boundary of the set, and finally the closure of the set.

Definition 6.67
Let $\xi = (X, \mathcal{T})$ be a topological space and $A \subset X$. A point $x \in X$ is called **interior point of** A if $A \in \mathcal{N}\{x\}$. The set of all interior points of A is called the **interior of** A and is denoted by $\mathrm{int}_\xi(A)$ or $\mathrm{int}_\xi A$. If the set X is evident from the context, we also write $\mathrm{int}_\mathcal{T}(A)$ or $\mathrm{int}_\mathcal{T} A$. If the topological space is evident from the context, we also write $\mathrm{int}(A)$, $\mathrm{int}\, A$, or A°. ■

The following is a characterization of the interior of a set.

Lemma 6.68
Let $(X, \mathcal{T})$ be a topological space, $A \subset X$, and $\mathcal{A} = \{U \in \mathcal{T} : U \subset A\}$. We have

$$A^\circ = \bigcup \mathcal{A} = \sup \mathcal{A} \in \mathcal{T}$$

where the supremum is with respect to the ordering $\subset$ on $\mathcal{P}(X)$. A° is the largest open set contained in A, i.e. it is the maximum of $\mathcal{A}$.

Proof. Exercise. □

Definition 6.69
Let $\xi = (X, \mathcal{T})$ be a topological space and $A \subset X$. A point $x \in X$ is called **accumulation point of** A if $(A \cap U) \setminus \{x\} \neq \emptyset$ for every $U \in \mathcal{N}\{x\}$. The set of all accumulation points of A is called the **derived set of** A and is denoted by $\mathrm{der}_\xi(A)$ or $\mathrm{der}_\xi A$. If the set X is evident from the context, we also write $\mathrm{der}_\mathcal{T}(A)$ or $\mathrm{der}_\mathcal{T} A$. If the topological space is evident from the context, we also write $\mathrm{der}(A)$, $\mathrm{der}\, A$, or A^d.
The union $A \cup A^d$ is called the **closure of** A and is denoted by $\mathrm{cl}_\xi(A)$ or $\mathrm{cl}_\xi A$. If the set X is evident from the context, we also write $\mathrm{cl}_\mathcal{T}(A)$ or $\mathrm{cl}_\mathcal{T} A$. If the topological space is evident from the context, we also write $\mathrm{cl}(A)$, $\mathrm{cl}\, A$, or $\overline{A}$. ■

The following result is a convenient characterization of the closure of a set.

Lemma 6.70
Let $(X, \mathcal{T})$ be a topological space, $\mathcal{C}$ the system of all closed sets, $A \subset X$, and $\mathcal{A} = \{B \in \mathcal{C} : A \subset B\}$. We have

$$\overline{A} = \bigcap \mathcal{A} = \inf \mathcal{A} \in \mathcal{C}$$

where the infimum is with respect to the ordering $\subset$ on $\mathcal{P}(X)$. $\overline{A}$ is the smallest closed set containing A, i.e. it is the minimum of $\mathcal{A}$.

Proof. To show the first equation, assume that $x \notin \bigcap \mathcal{A}$. Then there is $B \in \mathcal{C}$ with $A \subset B$ and $x \notin B$. It follows that $A \cap B^c = \emptyset$ and $x \in B^c$. Hence $x \notin A^d$. It is also clear that $x \notin A$. Therefore we have $x \notin \overline{A}$.
Conversely, assume that $x \notin \overline{A}$. Then there is $U \in \mathcal{N}\{x\}$ such that $A \cap U = \emptyset$. Hence there is $V \in \mathcal{T}$ such that $x \in V \subset U$. We have $A \cap V = \emptyset$. It follows that $A \subset V^c$, and hence $x \notin \bigcap \mathcal{A}$.
The second equation follows by Example 2.50.
To see the last claim, note that $\overline{A}$ is a lower bound and also a member of $\mathcal{A}$. □

Those points that belong to the closure of a set may be characterized in terms of the convergence of nets and filters. The following result is applied frequently in the sequel.

Theorem 6.71
Let $(X, \mathcal{T})$ be a topological space, $A \subset X$, and $x \in X$. The following statements are equivalent:

(i) $x \in \overline{A}$

(ii) $\forall U \in \mathcal{N}\{x\} \quad U \cap A \neq \emptyset$

(iii) There is a net (x_n) in A such that $x_n \to x$.

(iv) There is a filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $\mathcal{F} \to x$.

(v) There is a filter base $\mathcal{B}$ on X such that $\mathcal{B} \subset \mathcal{P}(A)$ and $\mathcal{B} \to x$.

(vi) $x \in \operatorname{adh} \mathcal{F}$ where $\mathcal{F} = \{F \subset X \,:\, A \subset F\}$

Proof. The equivalence of (i) and (ii) is a direct consequence of Definition 6.69. We now show the equivalence of (i), and (iii) to (vi).
To show that (i) implies (iii), let $x \in \overline{A}$. If $x \in A$, then we may choose a constant net $(x_n : n \in D)$ in A with $x_n = x$ $(n \in D)$. If $x \in A^d$, we may choose for each $U \in \mathcal{N}\{x\}$ a point $x_U \in A \cap U$. Since $(\mathcal{N}\{x\}, \supset)$ is a directed space, $(x_U : U \in \mathcal{N}\{x\})$ is a net. Moreover, $x_U \to x$.
To see that (iii) implies (iv), let (x_n) be a net in X with values in A such that $x_n \to x$. Further let $\mathcal{F}$ be the filter on X generated by (x_n). Then we clearly have $A \in \mathcal{F}$ and $\mathcal{F} \to x$.
To prove that (iv) implies (v), let $\mathcal{F}$ be a filter on X such that $A \in \mathcal{F}$ and $\mathcal{F} \to x$. Then $\mathcal{B} = \{F \cap A : F \in \mathcal{F}\}$ is a filter base on X. Moreover, we have $\mathcal{B} \subset \mathcal{P}(A)$ and $\mathcal{B} \to x$.
To show that (v) implies (vi), let $\mathcal{B}$ be a filter base on X such that $\mathcal{B} \subset \mathcal{P}(A)$ and $\mathcal{B} \to x$. Further let $U \in \mathcal{N}\{x\}$ and $F \in \mathcal{F}$. There exists $B \in \mathcal{B}$ such that $B \subset U$. It follows that $B \subset U \cap A \subset U \cap F$.
Finally, to see that (vi implies (i), let $x \in (\operatorname{adh} \mathcal{F}) \setminus A$ and $U \in \mathcal{N}\{x\}$. Since

$A \in \mathcal{F}$, we have $A \cap U \neq \emptyset$, and thus $(A \cap U)\backslash\{x\} \neq \emptyset$. It follows that $x \in A^d$. □

The following Theorem provides a characterization of the points of the derived set.

Theorem 6.72
Let $(X, \mathcal{T})$ be a topological space, $A \subset X$, and $x \in X$. Then the following statements are equivalent:

(i) $x \in A^d$

(ii) There is a net (x_n) in $A\backslash\{x\}$ such that $x_n \to x$.

(iii) There is a filter base $\mathcal{B}$ on X such that $\mathcal{B} \subset \mathcal{P}(A\backslash\{x\})$ and $\mathcal{B} \to x$.

Proof. We first show that (i) implies (ii). Let $x \in A^d$. It follows that $x \in (A\backslash\{x\})^d$. Since

$$(A\backslash\{x\})^d \subset \overline{A\backslash\{x\}}$$

there is a net (x_n) in $A\backslash\{x\}$ such that $x_n \to x$ by Theorem 6.71.
To see that (ii) implies (iii), let (x_n) be a net in X with values in $A\backslash\{x\}$ such that $x_n \to x$. Further let $\mathcal{F}$ be the filter on X generated by (x_n). Then

$$\mathcal{B} = \big\{(F \cap A)\backslash\{x\} : F \in \mathcal{F}\big\}$$

is a filter base on X. Moreover, we have $\mathcal{B} \subset \mathcal{P}(A\backslash\{x\})$ and $\mathcal{B} \to x$.
Finally, to prove that (iii) implies (i), let $\mathcal{B}$ be a filter base on X such that $\mathcal{B} \subset \mathcal{P}(A\backslash\{x\})$ and $\mathcal{B} \to x$. Further let $U \in \mathcal{N}\{x\}$. There exists $B \in \mathcal{B}$ such that $B \subset U$. It follows that $(A\backslash\{x\}) \cap U \neq \emptyset$. □

In the case of first countability, sequences can be used instead of nets in Theorems 6.71 and 6.72.

Lemma 6.73
Let $(X, \mathcal{T})$ be a topological space that is first countable, $A \subset X$, and $x \in X$. Then the following statements hold:

(i) $x \in \overline{A}$ iff there is a sequence (x_n) in A such that $x_n \to x$.

(ii) $x \in A^d$ iff there is a sequence (x_n) in $A \backslash \{x\}$ such that $x_n \to x$.

Proof. Exercise. □

For a pseudo-metric space there is a further characterization of the closure of a set.

Lemma 6.74
Given a pseudo-metric space (X, d) and a subset $A \subset X$, we have $\overline{A} = \{x \in X : \text{dist}(A, x) = 0\}$ where the closure is with respect to the pseudo-metric topology.

Proof. Exercise. □

Definition 6.75
Let $\xi = (X, \mathcal{T})$ be a topological space and $A \subset X$. A point $x \in X$ is called **boundary point of** A if $A \cap U \neq \emptyset$ and $(X \setminus A) \cap U \neq \emptyset$ for every $U \in \mathcal{N}\{x\}$. The set of all boundary points of A is called the **boundary of** A and is denoted by $\text{bound}_\xi(A)$ or $\text{bound}_\xi A$. If the set X is evident from the context, we also write $\text{bound}_\mathcal{T}(A)$ or $\text{bound}_\mathcal{T} A$. If the topological space is evident from the context, we also write $\text{bound}(A)$ or $\text{bound}\, A$, or ∂A. ■

We have the following characterizations of the boundary.

Lemma 6.76

Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The following statements hold:

(i) $x \in \partial A$ iff there is a net (x_n) in A such that $x_n \to x$ and a net (y_m) in A^c such that $y_m \to x$.

(ii) $\partial A = \overline{A} \cap \overline{A^c}$

Proof. Exercise. □

The next Lemma contains various results involving the interior, closure, derived set, and boundary of a set.

Lemma 6.77
Let $(X, \mathcal{T})$ be a topological space, $\mathcal{C}$ the system of all closed sets, and $A \subset X$. The following statements hold:

(i) $A^\circ \subset A$

(ii) $\partial A \in \mathcal{C}$

(iii) $A^\circ \cap \partial A = \emptyset$

(iv) $\overline{A} = A^\circ \cup \partial A$

(v) $A \in \mathcal{C} \iff \overline{A} = A \iff A^d \subset A \iff \partial A \subset A$

(vi) $A \in \mathcal{T} \iff A^\circ = A$

(vii) $\overline{A^c} = (A^\circ)^c$

(viii) $(A^\circ)^\circ = A^\circ$

(ix) $\overline{\overline{A}} = \overline{A}$

(x) $\partial(\partial A) \subset \partial A$

Proof. (i) follows by Definition 6.67.
(ii) follows by Lemma 6.76 (ii).
(iii) is obvious.
In order to see (iv), note that clearly $A^\circ \subset \overline{A}$. Moreover, we have $\partial A \subset \overline{A}$ by Lemma 6.76 (ii). Conversely, we have $A \subset A^\circ \cup \partial A$. Moreover, let $x \in A^d \setminus A$ if such a point exists. Then for every $U \in \mathcal{N}\{x\}$ we have $U \cap A \neq \emptyset$ by definition and $U \cap A^c \neq \emptyset$ since $x \in A^c$. Hence $x \in \partial A$.
The first equivalence of (v) follows by Lemma 6.70, the second by definition, and the third by (iv).
(vi) is obvious.

To show (vii), notice that

$$\begin{aligned}(A^\circ)^c &= \left(\bigcup\{U \in \mathcal{T} : U \subset A\}\right)^c \\ &= \bigcap\{U^c : U \in \mathcal{T}, U \subset A\} \\ &= \bigcap\{C \in \mathcal{C} : A^c \subset C\} = \overline{A^c}\end{aligned}$$

(viii) is a consequence of Lemma (6.68) and (vi).

(ix) follows by Lemma (6.70) and (v).

(x) follows by (ii) and (v). □

The interplay of intersection and union with interior, closure, and derived set is now investigated.

Lemma 6.78

Let $(X, \mathcal{T})$ be a topological space, $A, B \subset X$, and $A_i \subset X$ $(i \in I)$ where I is an index set. The following statements hold:

(i) $(A \cap B)^\circ = A^\circ \cap B^\circ$

(ii) $\left(\bigcap_{i\in I} A_i\right)^\circ \subset \bigcap_{i\in I} A_i^\circ$

(iii) $\overline{\bigcap_{i\in I} A_i} \subset \bigcap_{i\in I} \overline{A_i}$

(iv) $\left(\bigcap_{i\in I} A_i\right)^d \subset \bigcap_{i\in I} A_i^d$

(v) $\bigcup_{i\in I} A_i^\circ \subset \left(\bigcup_{i\in I} A_i\right)^\circ$

(vi) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(vii) $\bigcup_{i\in I} \overline{A_i} \subset \overline{\bigcup_{i\in I} A_i}$

(viii) $(A \cup B)^d = A^d \cup B^d$

(ix) $\bigcup_{i\in I} A_i^d \subset \left(\bigcup_{i\in I} A_i\right)^d$

Proof. (i), (ii), (v), and (ix) hold by definition.
To see (iii) and (iv), we define $B = \bigcap \mathcal{A}$. For every $A \in \mathcal{A}$ we have $B \subset A$ and therefore $\overline{B} \subset \overline{A}$ by Theorem 6.71, and $B^d \subset A^d$ by Theorem 6.72.
In order to see (vi), note that

$$\begin{aligned}\overline{A \cup B} &= \overline{(A^c \cap B^c)^c} = \big((A^c \cap B^c)^\circ\big)^c = \big((A^c)^\circ \cap (B^c)^\circ\big)^c \\ &= \big((A^c)^\circ\big)^c \cup \big((B^c)^\circ\big)^c = \overline{A} \cup \overline{B}\end{aligned}$$

by (i) and Lemma 6.77 (vii).
(vii) follows by Theorem 6.71.
To show (viii) notice that $(A \cup B)^d \supset A^d \cup B^d$ clearly holds. Conversely, let $x \in (A \cup B)^d$. Assume there are $U, V \in \mathcal{N}^{\text{open}}\{x\}$ such that $(U \cap A)\setminus\{x\} = \emptyset$ and $(V \cap B)\setminus\{x\} = \emptyset$. Thus we have $\big((U \cap V) \cap (A \cup B)\big) \setminus \{x\} = \emptyset$, which is a contradiction. □

The following Remarks and Examples demonstrate that stricter statements than those in Lemma 6.78 are generally not possible to achieve.

Example 6.79
In order to show that equality does generally not hold in Lemma 6.78 (ii), consider the standard ordering $<$ on $\mathbb{R}$ and the system of proper intervals $\mathcal{A} = \big\{\,]-n^{-1}, n^{-1}[\; : \, n \in \mathbb{N}, \, n > 0\big\}$. ■

Example 6.80
Generally equality does not hold in (iii) and (iv) even if I is finite. To see this consider the standard ordering $<$ on $\mathbb{R}$ and the system $\mathcal{A} = \{\,]-\infty, 0[\,, \,]0, \infty[\, \}$.
■

Example 6.81
Generally equality does not hold in (v) even if I is finite. To see this consider the standard ordering $<$ on $\mathbb{R}$ and the system $\mathcal{A} = \{\, [0,1]\,,\, [1,2]\,\}$. ■

Example 6.82
To see that equality does generally not hold in (vii) and (ix) consider the standard ordering $<$ on $\mathbb{R}$ and the system $\mathcal{A} = \big\{ \big[0, 1 - n^{-1}\big] \;:\; n \in \mathbb{N},\, n > 0 \big\}$. ■

A condition under which the reverse of Lemma 6.78 (vii) is true is provided in Lemma 7.25 below.
It is possible to characterize a topology on a given set X by expressing what it means to form the closure $\overline{A}$ of A for every $A \subset X$. This characterization is provided in Theorem 6.84 below. We begin with listing the relevant properties of the function that maps a set A on its closure with respect to a given topology.

Definition 6.83
Given a set X, a function $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is called **closure operator on** X if it has the following properties:

(i) $f(\emptyset) = \emptyset$

(ii) $A \subset f(A)$

(iii) $f(A \cup B) = f(A) \cup f(B)$

(iv) $f(f(A)) = f(A)$

■

The properties (i) to (iv) in Definition 6.83 are called Kuratowski axioms in the literature, cf. [Gaal].

Theorem 6.84
Let X be a set, f a closure operator on X, and $\mathcal{C} = \{A \subset X : f(A) = A\}$. The system $\mathcal{T} = \{A^c : A \in \mathcal{C}\}$ is the unique topology on X such that $\mathrm{cl}_{\mathcal{T}} A = f(A)$ for every $A \subset X$. Moreover, we have $\operatorname{ran} f = \mathcal{C}$.

Proof. $\mathcal{C}$ has properties (i) to (iii) in Lemma 5.12.

> [We have $\emptyset \in \mathcal{C}$ because of property (i) in Definition 6.83, and $X \in \mathcal{C}$ because of property (ii). Further, property (iii) implies that $A \cup B \in \mathcal{C}$ for every $A, B \in \mathcal{C}$. Finally let $\mathcal{A} \subset \mathcal{C}$ where $\mathcal{A} \neq \emptyset$. We clearly have $\bigcap \mathcal{A} \subset f\big(\bigcap \mathcal{A}\big)$ by property (ii). Moreover, for every $A \in \mathcal{A}$ we have $\bigcap \mathcal{A} \subset A$, and therefore $f\big(\bigcap \mathcal{A}\big) \subset f(A) = A$, and thus $f\big(\bigcap \mathcal{A}\big) \subset \bigcap \mathcal{A}$. Hence we obtain $f\big(\bigcap \mathcal{A}\big) = \bigcap \mathcal{A}$, and therefore $\bigcap \mathcal{A} \in \mathcal{C}$.]

Therefore $\mathcal{T}$ is a topology on X by Lemma 5.13, and $\mathcal{C}$ is the system of all $\mathcal{T}$-closed sets.
Now let $A \subset X$. We have $f(A) \in \mathcal{C}$ by Definition 6.83 (iv). It follows that

$$\mathrm{cl}_{\mathcal{T}} A = \bigcap \big\{B \in \mathcal{C} : A \subset B\big\} \subset f(A)$$

by Lemma 6.70 and Definition 6.83 (ii). Conversely, for every $B \in \mathcal{C}$, $A \subset B$ implies $f(A) \subset f(B) = B$. Hence we have $f(A) \subset \mathrm{cl}_{\mathcal{T}} A$. Thus we obtain $\mathrm{cl}_{\mathcal{T}} A = f(A)$.
To see the uniqueness, note that for a topology on X with the stated property, the system of all closed sets is precisely $\mathcal{C}$ by Lemma 6.77 (v).
The last claim clearly holds. $\square$

Similarly to the closure operator we introduce the notion of interior operator.

Definition 6.85
Given a set X, a function $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is called **interior operator on** X if it has the following properties:

(i) $f(X) = X$

(ii) $f(A) \subset A$

(iii) $f(A \cap B) = f(A) \cap f(B)$

(iv) $f(f(A)) = f(A)$

■

There is a duality between closure and interior operators as follows.

Lemma 6.86
Let X be a set, $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ a map, and $g : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ the map defined by $g(A) = \big(f(A^c)\big)^c$. We have $f(A) = \big(g(A^c)\big)^c$ for every $A \subset X$. Moreover, f is a closure operator on X iff g is an interior operator on X.

Proof. Exercise. □

It follows that a given interior operator on X defines a certain topology on X.

Theorem 6.87
Let X be a set, $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ an interior operator on X, and $\mathcal{T} = \{U \subset X : f(U) = U\}$. The system $\mathcal{T}$ is the unique topology on X such that $f(A) = \mathrm{int}_{\mathcal{T}} A$ for every $A \subset X$. Moreover, we have $\mathrm{ran}\, f = \mathcal{T}$.

Proof. Let g be the closure operator on X defined by $g(A) = \big(f(A^c)\big)^c$ for every $A \subset X$ (cf. Lemma 6.86), and $\mathcal{T}_g = \{A^c : A \subset X, g(A) = A\}$. Then $\mathcal{T}_g$

is the unique topology on X such that $\mathrm{cl}_{\mathcal{T}(g)} A = g(A)$ for every $A \subset X$ by Theorem 6.84. We have

$$\mathcal{T}_g = \{U \subset X : g(U^c) = U^c\} = \mathcal{T}$$

which shows that $\mathcal{T}$ is a topology on X.
Moreover, for every $A \subset X$ we have

$$f(A) = \big(g(A^c)\big)^c = \big(\mathrm{cl}_{\mathcal{T}}(A^c)\big)^c = \mathrm{int}_{\mathcal{T}} A$$

by Lemma 6.77 (vii). Since the range of g is the system of all closed sets, the range of f is $\mathcal{T}$.
To see the uniqueness, notice that a topology on X with the stated property is equal to $\mathcal{T}$ by Lemma 6.77 (vi). □

Using closure and interior we obtain further characterizations of continuity of a function.

Theorem 6.88
Given two topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and a map $f : X \longrightarrow Y$, the following statements are equivalent:

(i) f is continuous.

(ii) $\forall A \subset X \quad f\left[\overline{A}\right] \subset \overline{f[A]}$

(iii) $\forall B \subset Y \quad \overline{f^{-1}[B]} \subset f^{-1}\left[\overline{B}\right]$

(iv) $\forall B \subset Y \quad f^{-1}[B^\circ] \subset \left(f^{-1}[B]\right)^\circ$

Proof. The fact that (i) implies (ii) follows by Theorems 6.47 and 6.71.
To prove that (ii) implies (iii), let $B \subset Y$ and $A = f^{-1}[B]$. Then we have $f[A] \subset B$, and thus $f\left[\overline{A}\right] \subset \overline{f[A]} \subset \overline{B}$. It follows that

$$\overline{f^{-1}[B]} = \overline{A} \subset f^{-1}\left[f\left[\overline{A}\right]\right] \subset f^{-1}\left[\overline{B}\right]$$

To see that (iii) implies (iv), let $B \subset Y$. We have

$$\begin{aligned} f^{-1}[B^\circ] &= f^{-1}\left[\left(\overline{B^c}\right)^c\right] = \left(f^{-1}\left[\overline{B^c}\right]\right)^c \subset \left(\overline{f^{-1}[B^c]}\right)^c \\ &= \left(\overline{(f^{-1}[B])^c}\right)^c = \left(f^{-1}[B]\right)^\circ \end{aligned}$$

To see that (iv) implies (i), let $x \in X$ and $U \in \mathcal{N}\{f(x)\}$. Then we have $f(x) \in U^\circ$. It follows that

$$x \in f^{-1}[U^\circ] \subset \left(f^{-1}[U]\right)^\circ \subset f^{-1}[U]$$

and thus $f^{-1}[U] \in \mathcal{N}\{x\}$. □

In Section 6.1 we have investigated the convergence of sequences with respect to comparable topologies on the same set, see Lemmas 6.14 and 6.90. In particular, we have found that if a topology $\mathcal{T}_1$ is finer than a topology $\mathcal{T}_2$, then $\mathcal{T}_1$ is also sequentially stronger than $\mathcal{T}_2$. As shown now the converse implication holds if the topologies are countable. We also establish similar results for the convergence properties of filters and nets. In the proofs we follow [Wilansky].

Proposition 6.89
Let X be a set and $\mathcal{T}_1$, $\mathcal{T}_2$ two topologies on X. $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ iff $\mathrm{cl}_{\mathcal{T}(1)} A \subset \mathrm{cl}_{\mathcal{T}(2)} A$.

Proof. For $i \in \{1, 2\}$ let $\mathcal{C}_i$ be the system of all $\mathcal{T}_i$-closed sets.
First assume that $\mathcal{T}_2 \subset \mathcal{T}_1$. It follows that $\mathcal{C}_2 \subset \mathcal{C}_1$ and therefore

$$\begin{aligned} \mathrm{cl}_{\mathcal{T}(1)} A &= \bigcap \{B \in \mathcal{C}_1 : A \subset B\} \\ &\subset \bigcap \{B \in \mathcal{C}_2 : A \subset B\} = \mathrm{cl}_{\mathcal{T}(2)} A \end{aligned}$$

by Lemma 6.70.

Now assume that $\mathrm{cl}_{\mathcal{T}(1)} A \subset \mathrm{cl}_{\mathcal{T}(2)} A$ for every $A \subset X$. Let $A \in \mathcal{T}_2$. Then we have $A^c \in \mathcal{C}_2$. Moreover, by assumption we have $\mathrm{cl}_{\mathcal{T}(1)} A^c \subset \mathrm{cl}_{\mathcal{T}(2)} A^c = A^c$. It follows that $\mathrm{cl}_{\mathcal{T}(1)} A^c = A^c$. Thus we have $A^c \in \mathcal{C}_1$, and hence $A \in \mathcal{T}_1$. □

Lemma 6.90

Let X be a set, $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies on X where $\mathcal{T}_1$ is first countable. $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ iff $\mathcal{T}_1$ is sequentially stronger than $\mathcal{T}_2$.

Proof. First assume that $\mathcal{T}_1$ is sequentially stronger than $\mathcal{T}_2$. Then we have $\mathrm{cl}_{\mathcal{T}(1)} A \subset \mathrm{cl}_{\mathcal{T}(2)} A$ by Lemma 6.73 (ii). It follows that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ by Proposition 6.89.
The converse follows by Lemma 6.14. □

Theorem 6.91

Let X be a set, and $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies on X. Then the following statements are equivalent:

(i) $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

(ii) For every net (x_n) in X and every $x \in X$, $x_n \to x$ with respect to $\mathcal{T}_1$ implies $x_n \to x$ with respect to $\mathcal{T}_2$.

(iii) For every filter $\mathcal{F}$ in X and every $x \in X$, $\mathcal{F} \to x$ with respect to $\mathcal{T}_1$ implies $\mathcal{F} \to x$ with respect to $\mathcal{T}_2$.

Proof. (i) implies (ii) by Lemma 5.97.
To see that (ii) implies (i), assume that (ii) holds. Let $A \subset X$. We have $\mathrm{cl}_{\mathcal{T}(1)} A \subset \mathrm{cl}_{\mathcal{T}(2)} A$.

> [Let $x \in \mathrm{cl}_{\mathcal{T}(1)} A$. We may choose a net (x_n) in A such that $x_n \to x$ with respect to $\mathcal{T}_1$ by Theorem 6.71. Then we have $x_n \to x$ with respect to $\mathcal{T}_2$ by assumption. It follows that $x \in \mathrm{cl}_{\mathcal{T}(2)} A$.]

Now (i) follows by Proposition 6.89.

The equivalence of (ii) and (iii) follows by Lemma 6.39 and by Lemma and Definition 6.40. □

6.6 Separability

In this Section we provide a brief discussion of separability.

Definition 6.92

Given a topological space $(X, \mathcal{T})$ and $A \subset X$, A is called **dense in** X if $\overline{A} = X$.
■

Order dense sets as introduced in Definition 2.29 in the context of ordered spaces are related to dense sets as follows.

Lemma 6.93

Let $(X, \prec)$ be a pre-ordered space where $\prec$ has full field and the interval intersection property. Further let $A \subset X$ be an order dense subset. Then A is dense in X with respect to the interval topology.

Proof. We define

$$\begin{aligned} \mathcal{S} &= \big\{ \,]{-\infty}, x[\; , \;]x, \infty[\; : \; x \in X \big\}, \\ \mathcal{A} &= \big\{ \,]x, y[\; : \; x, y \in X, \; x \prec y \big\}, \\ \mathcal{B} &= \mathcal{S} \cup \mathcal{A} \cup \{\emptyset\} \end{aligned}$$

The system $\mathcal{B}$ is a base for the interval topology by Lemma 5.99. Let $x \in X$ and $U \in \mathcal{N}\{x\}$. Then there is $B \in \mathcal{B}$ such that $x \in B \subset U$. Moreover we have $B \cap A \neq \emptyset$ since A is order dense. Thus $x \in \overline{A}$ by Theorem 6.71. □

Definition 6.94
A topological space is called **separable** if it contains a countable subset that is dense in X. ■

Example 6.95
Let $<$ be the ordering on $\mathbb{R}_+$ as defined in Lemma and Definition 4.13. It has full field and the interval intersection property, cf. Remark 5.103. Moreover $\mathbb{D}_+$ is order dense in $\mathbb{R}_+$ by Lemma 4.15. Thus $\mathbb{D}_+$ is dense in $\mathbb{R}_+$ with respect to the standard topology by Lemma 6.93. Since $\mathbb{D}_+$ is countable by Corollary 4.3, $(\mathbb{R}_+, <)$ is separable. Similarly it can be shown that $(\mathbb{R}, <)$ is separable where $<$ is the standard ordering. ■

We now briefly examine how the properties of first and second countability of a topological space are related to separability.

Lemma 6.96
A topological space $(X, \mathcal{T})$ that is second countable is separable.

Proof. Let $\mathcal{B}$ be a countable base for $\mathcal{T}$. For each $B \in \mathcal{B}$ with $B \neq \emptyset$ we may choose a point $x_B \in B$. We define $A = \{x_B : B \in \mathcal{B}\}$. Let $x \in X$ and $U \in \mathcal{N}\{x\}$. Then $U \cap A \neq \emptyset$. It follows that $x \in \overline{A}$ by Theorem 6.71. □

Notice that a separable space need not even be first countable as shown in the following Example that we take from [Steen].

Example 6.97

Let X be an infinite set that is not countable and $\mathcal{T}_{\mathrm{cf}}$ the cofinite topology, cf. Lemma and Definition 5.10 (iii).

Let $A \subset X$ be countable and infinite. Then $U \cap A \neq \emptyset$ for every $U \in \mathcal{N}\{x\}$ and every $x \in X$. Hence $(X, \mathcal{T}_{\mathrm{cf}})$ is separable by Theorem 6.71.

Now assume that $(X, \mathcal{T}_{\mathrm{cf}})$ is first countable. Then there is a point $x \in X$ and a countable neighborhood base $\mathcal{B}$ of x. We define $D = \bigcap \mathcal{B}$. We have $D = \{x\}$ by the definition of $\mathcal{T}_{\mathrm{cf}}$, and therefore $X \backslash \{x\} = \bigcup \{B^c : B \in \mathcal{B}\}$. Since B^c is finite for every $B \in \mathcal{B}$, it follows that $X \backslash \{x\}$ is countable by Lemma 3.70, which is a contradiction. ■

Finally the following results holds for pseudo-metrizable spaces.

Theorem 6.98

A pseudo-metrizable topological space is separable iff it is second countable.

Proof. Let $\xi = (X, \mathcal{T})$ be a pseudo-metrizable topological space and d a pseudo-metric on X that generates $\mathcal{T}$. Assume that ξ is separable. Let A be a countable set that is dense in X and $R = \mathbb{D}_{+} \backslash \{0\}$. Then the system

$$\mathcal{B} = \left\{ B(x, r) \, : \, x \in A, \, r \in R \right\}$$

is a countable base for $\mathcal{T}$ by Lemma 3.69.

[Let $U \in \mathcal{T}$ and $x \in U$. We may choose $y \in X$ and $r \in R$ such that $x \in B(y,r) \subset U$ by the definition of the pseudo-metric topology. We define $\varepsilon = r - d(x,y)$. $\mathcal{T}$ is first countable by Theorem 5.127. Hence there is a sequence (x_n) in A such that $x_n \to x$ by Lemma 6.73 (i). We may choose $m \in \mathbb{N}$ such that $d(x_m, x) < \varepsilon/2$. It follows that $x \in B$ where $B = B(x_m, \varepsilon/2)$. For every $z \in B$ we have

$$d(y,z) \;\leq\; d(y,x) + d(x,x_m) + d(x_m,z) \;<\; d(x,y) + \varepsilon \;=\; r$$

and hence $B \subset B(y,r)$.]

The converse is true by Lemma 6.96. □

Chapter 7

Generated topologies

In this Chapter we explore how a topology on a given set can be defined by means of already specified topologies on—generally different—sets and functions between the old and the new sets. Obviously there are two possibilities: the new topology may be generated on the domain of the functions by given topologies on their ranges (so-called inverse image topology, see Section 7.1) or generated on their range by given topologies on their domains (so-called direct image topology, see Section 7.4). In Sections 7.2 and 7.3 we analyse two particular cases of inverse image topologies in detail: subspace topologies and product topologies. Section 7.5 is devoted to quotient topologies which are special cases of direct image topologies.

7.1 Inverse image topology

Lemma and Definition 7.1

Given a set X, topological spaces $(Y_i, \mathcal{T}_i)$ $(i \in I)$, where I is an index set, and functions $f_i : X \longrightarrow Y_i$ $(i \in I)$, the system $\mathcal{S} = \bigcup_{i \in I} f_i^{-1} [\![\mathcal{T}_i]\!]$ is a topological subbase on X. The topology on X that is generated by $\mathcal{S}$ is called **inverse image topology** or the **topology generated by** $F = \{(f_i, \mathcal{T}_i) : i \in I\}$ and denoted by $\tau(F)$. It is the coarsest topology $\mathcal{T}$ on X such that f_i is $\mathcal{T}$-$\mathcal{T}_i$-continuous for every $i \in I$. We also use the notation $x_i = f_i(x)$ for $x \in X$ and $i \in I$.

Proof. $\mathcal{S}$ is a topological subbase on X by Lemma 5.38. We denote by $\mathscr{A}$ the set of all topologies $\mathcal{T}$ on X such that f_i is $\mathcal{T}$-$\mathcal{T}_i$-continuous for every $i \in I$. Then we have $\tau(F) \in \mathscr{A}$ by Theorem 6.49. Moreover, for every $\mathcal{T} \in \mathscr{A}$ we have $\mathcal{S} \subset \mathcal{T}$. Hence $\tau(F)$ is the coarsest member of $\mathscr{A}$ by Lemma 5.38. □

Lemma 7.2
With definitions as in Lemma and Definition 7.1, let $\mathcal{S}_i$ be a subbase for $\mathcal{T}_i$ for every $i \in I$. The system $\mathcal{R} = \bigcup_{i\in I} f_i^{-1} [\![\mathcal{S}_i]\!]$ is a subbase for $\tau(F)$.

Proof. First note that $\mathcal{R}$ is a topological subbase on X by Lemma 5.38. We denote by $\mathcal{T}$ the topology generated by $\mathcal{R}$. Now, $\mathcal{R} \subset \mathcal{S}$ implies $\mathcal{T} = \Theta\Psi(\mathcal{R}) \subset \Theta\Psi(\mathcal{S}) = \tau(F)$. Moreover, for every $i \in I$, f_i is $\mathcal{T}$-$\mathcal{T}_i$-continuous by Theorem 6.49. Since $\tau(F)$ is the coarsest such topology, we have $\tau(F) \subset \mathcal{T}$. □

Corollary 7.3
With definitions as in Lemma and Definition 7.1, let $\mathcal{C}$ be the system of all $\tau(F)$-closed sets and, for each $i \in I$, let $\mathcal{E}_i$ be a subbase for the $\mathcal{T}_i$-closed sets. Then $\bigcup_{i\in I} f_i^{-1} [\![\mathcal{E}_i]\!]$ is a subbase for $\mathcal{C}$.

Proof. This is a consequence of Lemma 7.2 by complementation. □

The following is an important special case of Lemma and Definition 7.1 and Lemma 7.2.

Corollary 7.4
Let X be a set and I an index set. For each $i \in I$, let $\mathcal{T}_i$ be a topology on X and $\mathcal{S}_i$ a subbase for $\mathcal{T}_i$. Further let $F = \{(\mathrm{id}_X, \mathcal{T}_i) : i \in I\}$. $\bigcup_{i\in I} \mathcal{S}_i$ is a subbase for $\tau(F)$. $\tau(F)$ is the supremum of $\{\mathcal{T}_i : i \in I\}$ in the ordered space $(\mathscr{T}(X), \subset)$, i.e. it is the coarsest topology on X that is finer than $\mathcal{T}_i$ for every $i \in I$.

Proof. Exercise. □

Another relevant special case is that of a topology generated by a single map. We have the following result.

Corollary 7.5

Let X be a set, $(Y, \mathcal{T})$ a topological space, $\mathcal{B}$ a base for $\mathcal{T}$, $\mathcal{S}$ a subbase for $\mathcal{T}$, $\mathcal{C}$ the system of all $\mathcal{T}$-closed sets, $\mathcal{D}$ a base for $\mathcal{C}$, $\mathcal{E}$ a subbase for $\mathcal{C}$, $f : X \longrightarrow Y$ a map, and $\mathcal{T}_X = \tau\left(\{(f, \mathcal{T})\}\right)$. Then the following statements hold:

(i) $\mathcal{T}_X = f^{-1}[\![\mathcal{T}]\!]$

(ii) $f^{-1}[\![\mathcal{B}]\!]$ is a base for $\mathcal{T}_X$.

(iii) $f^{-1}[\![\mathcal{S}]\!]$ is a subbase for $\mathcal{T}_X$.

(iv) $f^{-1}[\![\mathcal{C}]\!]$ is the system of all $\mathcal{T}_X$-closed sets.

(v) $f^{-1}[\![\mathcal{D}]\!]$ is a base for $f^{-1}[\![\mathcal{C}]\!]$.

(vi) $f^{-1}[\![\mathcal{E}]\!]$ is a subbase for $f^{-1}[\![\mathcal{C}]\!]$.

Proof. Exercise. □

Example 7.6

Let $m, n \in \mathbb{N}$ with $0 < m < n$, $\mathcal{T}_m$ and $\mathcal{T}_n$ the standard topologies on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and $x \in \mathbb{R}^{n-m}$. Further we define the function

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n\,, \quad f(z) = (z, x)$$

where we have identified $\mathbb{R}^n$ and $\mathbb{R}^m \times \mathbb{R}^{n-m}$. Then we have $\tau(\{(f, \mathcal{T}_n)\}) = \mathcal{T}_m$.
■

The following Lemma shows how the generation of topologies behaves under the composition of functions.

Lemma 7.7
Let X and Y_i $(i \in I)$ be sets where I is an index set, $(Z_j, \mathcal{T}_j)$ $(j \in J_i,\, i \in I)$ topological spaces where J_i $(i \in I)$ are distinct index sets, and $f_i : X \longrightarrow Y_i$ $(i \in I)$ and $g_j : Y_i \longrightarrow Z_j$ $(i \in I,\, j \in J_i)$ maps. For every $i \in I$, we define the following topology on Y_i :

$$\mathcal{T}_i = \tau\left(\{(g_j, \mathcal{T}_j) \,:\, j \in J_i\}\right)$$

We have

$$\tau\left(\{(f_i, \mathcal{T}_i) \,:\, i \in I\}\right) = \tau\left(\{(g_j \circ f_i, \mathcal{T}_j) \,:\, i \in I,\, j \in J_i\}\right)$$

Proof. This is a consequence of Lemma 7.2. □

Theorem 7.8
Let $(X, \mathcal{T})$ be a topological space, $(Y_i, \mathcal{T}_i)$ $(i \in I)$ topological spaces where I is an index set, $f_i : X \longrightarrow Y_i$ $(i \in I)$ functions, and $F = \{(f_i, \mathcal{T}_i) \,:\, i \in I\}$. The following statements are equivalent:

(i) $\mathcal{T} = \tau(F)$

(ii) For every topological space $(Z, \mathcal{T}_Z)$ and every function $g : Z \longrightarrow X$, g is $\mathcal{T}_Z$-$\mathcal{T}$-continuous iff $f_i \circ g$ is $\mathcal{T}_Z$-$\mathcal{T}_i$-continuous for every $i \in I$.

Proof. To see that (i) implies (ii), assume that $\tau(F) = \mathcal{T}$. Then f_i is $\mathcal{T}$-$\mathcal{T}_i$-continuous for every $i \in I$. Further let $(Z, \mathcal{T}_Z)$ be a topological space and $g : Z \longrightarrow X$ a map. Then the continuity of g implies the continuity of $f_i \circ g$ for every $i \in I$ by Lemma 6.59. Conversely, for every $i \in I$ let $\mathcal{S}_i$ be a subbase for $\mathcal{T}_i$. Then we have $g^{-1}\left[f_i^{-1}[S]\right] \in \mathcal{T}_Z$ for every $i \in I$ and $S \in \mathcal{S}_i$ by the continuity of $f_i \circ g$. Since $\left\{f_i^{-1}[S] \,:\, i \in I,\, S \in \mathcal{S}_i\right\}$ is a subbase of $\mathcal{T}$, the continuity of g follows.

To show that (ii) implies (i), it is enough to show that the topology $\mathcal{T}$ is uniquely specified by property (ii). Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X such that (ii) is satisfied in both cases. Now let $Z = X$ and $g = \mathrm{id}_X$. Since g is $\mathcal{T}_m$-$\mathcal{T}_m$-continuous for $m \in \{1, 2\}$, it follows that f_i is $\mathcal{T}_m$-$\mathcal{T}_i$-continuous for $m \in \{1, 2\}$ and $i \in I$. Thus g is $\mathcal{T}_1$-$\mathcal{T}_2$-continuous and $\mathcal{T}_2$-$\mathcal{T}_1$-continuous, and hence $\mathcal{T}_1 = \mathcal{T}_2$. □

Property (ii) in Theorem 7.8 is often called a universal property. Below we encounter other universal properties a topological or a uniform space may have.

Theorem 7.9

With definitions as in Lemma and Definition 7.1, let $(x_n : n \in D)$ be a net in X, $\mathcal{F}$ a filter on X, and $x \in X$. The following statements hold:

(i) $x_n \to x \iff \forall i \in I \quad f_i(x_n) \to f_i(x)$

(ii) $\mathcal{F} \to x \iff \forall i \in I \quad f_i[\![\mathcal{F}]\!] \to f_i(x)$

Proof. To see (i), assume that $f_i(x_n) \to f_i(x)$ for every $i \in I$ and let $U \in \mathcal{N}\{x\}$. Then there is $\mathcal{G} \sqsubset \mathcal{S}$ such that $\mathcal{G} \neq \emptyset$ and $x \in \bigcap \mathcal{G} \subset U$. Now let $G \in \mathcal{G}$. Further let $i \in I$ and $V \in \mathcal{T}_i$ such that $G = f_i^{-1}[V]$. Thus there is $N_G \in D$ such that $f_i(x_n) \in V$ for every $n \geq N_G$ by assumption. We may choose such N_G for each $G \in \mathcal{G}$. It follows that $x_n \in U$ for $n \geq N$ where N is the maximum of $\{N_G : G \in \mathcal{G}\}$. The converse follows by Theorem 6.49.
To see (ii), assume that $f_i[\![\mathcal{F}]\!] \to f_i(x)$ for every $i \in I$. Let (y_m) be the net generated by $\mathcal{F}$, $U \in \mathcal{N}\{x\}$, and $i \in I$. There is $F \in \mathcal{F}$ such that $f_i[F] \subset U$ by assumption. Now let $M = (y, F)$ with $y \in F$. It follows that $f_i(y_m) \in U$ for every $m \geq M$. Hence we have $y_m \to x$ by (i), and thus $\mathcal{F} \to x$. Again the reverse implication follows by Theorem 6.49. □

7.2 Topological subspace

In this Section a special case of inverse image topologies is analyzed, viz. topological subspaces.

Lemma and Definition 7.10
Given a topological space $\xi = (X, \mathcal{T})$ and a subset $A \subset X$, the system $\{U \cap A : U \in \mathcal{T}\}$ is a topology on A. It is called the **relative topology on** A and denoted by $\mathcal{T} \,|\, A$. The pair $(A, \mathcal{T} \,|\, A)$ is called **topological subspace of** ξ, or short **subspace of** ξ, and denoted by $\xi \,|\, A$. A member of $\mathcal{T} \,|\, A$ is also called **open in** A whereas a member of $\mathcal{T}$ is also called **open in** X. Furthermore a $(\mathcal{T} \,|\, A)$-closed set is also called **closed in** A, and a $\mathcal{T}$-closed set is also called **closed in** X.

Proof. Exercise. □

Definition 7.11
Given a set X and a subset $A \subset X$, the map $j : A \longrightarrow X$, $j(x) = x$ $(x \in A)$, is called **inclusion**. We also write $j : A \hookrightarrow X$. ■

Lemma 7.12

Let $\xi = (X, \mathcal{T})$ be a topological space, $\alpha = (A, \mathcal{T}_A)$ a subspace of ξ, and $\mathcal{C}$ and $\mathcal{C}_A$ their respective systems of closed sets. Further let $\mathcal{B}$ be a base for $\mathcal{T}$, $\mathcal{S}$ a subbase for $\mathcal{T}$, $\mathcal{D}$ a base for the closed sets in X, $\mathcal{E}$ a subbase for the closed sets in X, $j : A \hookrightarrow X$ the inclusion, $x \in A$, and $B \subset A$. Then the following statements hold:

(i) We have $\mathcal{T}_A = j^{-1}[\![\mathcal{T}]\!]$. $\mathcal{T}_A$ is generated by $\{(j, \mathcal{T})\}$, and j is $\mathcal{T}_A$-$\mathcal{T}$-continuous.

(ii) The system $j^{-1}[\![\mathcal{B}]\!]$ is a base for $\mathcal{T}_A$.

(iii) The system $j^{-1}[\![\mathcal{S}]\!]$ is a subbase for $\mathcal{T}_A$.

(iv) Let $(Y, \mathcal{T}_Y)$ be a topological space and $g : Y \longrightarrow A$ a map. Then g is continuous iff $j \circ g$ is continuous.

(v) $\{A \cap C : C \in \mathcal{C}\} = j^{-1}[\![\mathcal{C}]\!] = \mathcal{C}_A$

(vi) The system $j^{-1}[\![\mathcal{D}]\!]$ is a base for $\mathcal{C}_A$.

(vii) The system $j^{-1}[\![\mathcal{E}]\!]$ is a subbase for $\mathcal{C}_A$.

(viii) $\big\{A \cap U : U \in \mathcal{N}_\xi\{x\}\big\} = j^{-1}[\![\mathcal{N}_\xi\{x\}]\!] = \mathcal{N}_\alpha\{x\}$

(ix) $\mathrm{cl}_\alpha B = \mathrm{cl}_\xi(B) \cap A$

(x) $\mathrm{int}_\xi B \subset \mathrm{int}_\alpha B$

(xi) $\mathrm{bound}_\alpha B \subset \mathrm{bound}_\xi B$

Proof. (i) is obvious.

(ii), (iii), and (v) to (vii are consequences of (i) and Corollary 7.5.

(iv) follows by (i) and Theorem 7.8.

The first equation in (viii) follows by definition of j. Moreover we have

$$j^{-1}\left[\!\left[\mathcal{N}_\xi\{x\}\right]\!\right] \subset \mathcal{N}_\alpha\{x\}$$

by the continuity of j. Conversely, let $U \in \mathcal{N}_\alpha\{x\}$. We may choose $V \in \mathcal{T}_A$ with $x \in V \subset U$, and $W \in \mathcal{T}$ with $V = W \cap A$. Let $R = W \cup U$. Then $R \in \mathcal{N}_\xi\{x\}$, and $j^{-1}[R] = U$.
(ix) follows by (v) and Lemma 6.70 as follows:

$$\begin{aligned} \mathrm{cl}_\alpha B &= \bigcap \left\{C \in \mathcal{C}_A \,:\, B \subset C\right\} \\ &= \bigcap \left\{D \cap A \,:\, B \subset D,\, D \in \mathcal{C}\right\} \\ &= \bigcap \left\{D \in \mathcal{C} \,:\, B \subset D\right\} \cap A \;=\; \mathrm{cl}_\xi(B) \cap A \end{aligned}$$

To see (x), notice that a set $U \subset A$ that is open in X is also open in A. Therefore we have:

$$\begin{aligned} \mathrm{int}_\xi B &= \bigcup \left\{U \in \mathcal{T} \,:\, U \subset B\right\} \\ &\subset \bigcup \left\{U \in \mathcal{T}_A \,:\, U \subset B\right\} \;=\; \mathrm{int}_\alpha B \end{aligned}$$

Finally, in order to prove (xi), let $x \in \mathrm{bound}_\alpha B$, $U \in \mathcal{N}_\xi^{\mathrm{open}}\{x\}$, and $V = U \cap A$. It follows that $V \in \mathcal{N}_\alpha\{x\}$, and thus $V \cap B \neq \emptyset$ and $V \cap (A \setminus B) \neq \emptyset$. Therefore we have $U \cap B \neq \emptyset$ and $U \cap (X \setminus B) \neq \emptyset$. It follows that $x \in \mathrm{bound}_\xi B$. □

Remark 7.13

Let $\xi = (X, \mathcal{T})$ be a topological space and $B \subset A \subset X$. If B is open in X, then B is open in A by definition of the relative topology. If B is closed in X, then B is closed in A by Lemma 7.12 (v). The converse of both implications is generally not true. However, it is true under additional assumptions, cf. Lemmas 7.19 and 7.20. ■

Remark 7.14

Let $\xi = (X, \mathcal{T})$ be a topological space and $B \subset A \subset X$. We have $\xi \,|\, B = (\xi \,|\, A) \,|\, B$, which is a consequence of Definition 7.10. It also follows by Lemmas 7.7 and 7.12 (i). ■

Lemma 7.15

Let $\mathcal{T}$ and $\mathcal{T}_+$ be the standard topologies on $\mathbb{R}$ and $\mathbb{R}_+$, respectively. We have $\mathcal{T}_+ = \mathcal{T} \,|\, \mathbb{R}_+$. Further let $A \subset \mathbb{R}_+$. Then $\mathcal{T}_+|A = \mathcal{T}\,|A$.

Proof. To see the first claim note that

$$\mathcal{S} = \big\{\,]{-\infty}, x[\ ,\ ]x, \infty[\ :\ x \in \mathbb{R} \big\}$$

is a subbase for $\mathcal{T}$ and

$$\mathcal{S}_+ = \big\{\,]{-\infty}, x[\ ,\ ]x, \infty[\ :\ x \in \mathbb{R}_+ \big\} \cup \{\mathbb{R}_+\}$$

is a subbase for $\mathcal{T}_+$. These two systems are related by $j^{-1}\,[\![\mathcal{S}]\!] = \mathcal{S}_+$ where $j : \mathbb{R}_+ \hookrightarrow \mathbb{R}$ is the inclusion. The claim follows by Theorem 7.12 (iii).
Now the second claim follows by Remark 7.14. □

The following three Lemmas show that the concept of topological subspace does not lead to any complications when considering convergent nets or filters and continuous functions.

Lemma 7.16

Let $\xi = (X, \mathcal{T})$ be a topological space, $(A, \mathcal{A})$ a subspace of ξ, (x_n) a net in A, and $x \in A$. Then $x_n \to x$ with respect to $\mathcal{A}$ iff $x_n \to x$ with respect to $\mathcal{T}$.

Proof. This is a special case of Theorem 7.9 (i). □

Lemma 7.17
Let $\xi = (X, \mathcal{T})$ be a topological space, $(A, \mathcal{A})$ a subspace of ξ, $x \in A$, and $\mathcal{G}$ a filter on A. Then $\mathcal{G}$ is a filter base for a filter on X, say $\mathcal{F}$. Moreover the following statements are equivalent:

(i) $\mathcal{G} \to x$ with respect to $\mathcal{A}$

(ii) $\mathcal{G} \to x$ with respect to $\mathcal{T}$, i.e. $\mathcal{G}$ is considered as filter base on X

(iii) $\mathcal{F} \to x$ with respect to $\mathcal{T}$

Proof. The equivalence of (i) and (ii) is a special case of Theorem 7.9 (ii). The equivalence of (ii) and (iii) follows by definition. □

Lemma 7.18
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, $(A, \mathcal{T}_A)$ a subspace of $(X, \mathcal{T}_X)$, $(B, \mathcal{T}_B)$ a subspace of $(Y, \mathcal{T}_Y)$, and $f : X \longrightarrow Y$ a map with $f[X] \subset B$. Further we define the map $g : X \longrightarrow B$, $g(x) = f(x)$. The following statements hold:

(i) If f is continuous, then $f \mid A$ is continuous.

(ii) f is continuous iff g is continuous.

Proof. Exercise. □

The following two Lemmas state under which conditions the reverse implications of Remark 7.13 are true.

Lemma 7.19
Let $\xi = (X, \mathcal{T})$ be a topological space, A a closed subset of X, $\alpha = \xi \,|\, A$, and $B \subset A$. The following statements hold:

(i) B is closed in X iff it is closed in A.

(ii) $\mathrm{cl}_\alpha B \;=\; \mathrm{cl}_\xi B$

Proof. (i) follows by Lemma 7.12 (v) and Remark 7.13.
To see (ii), notice that $\mathrm{cl}_\xi B \;\subset\; \mathrm{cl}_\xi A \;=\; A$. Thus we have $\mathrm{cl}_\alpha B \;=\; \mathrm{cl}_\xi B$ by Lemma 7.12 (ix). □

Lemma 7.20
Let $\xi = (X, \mathcal{T})$ be a topological space, $A \in \mathcal{T}$, $\alpha = \xi \,|\, A$, and $B \subset A$. The following statements hold:

(i) B is open in X iff it is open in A.

(ii) $\mathrm{int}_\alpha B \;=\; \mathrm{int}_\xi B$

(iii) $\mathrm{bound}_\alpha B \;=\; \mathrm{bound}_\xi(B) \cap A$

Proof. (i) is obvious.
To show (ii), let $x \in \mathrm{int}_\alpha B$ and U open in A with $x \in U \subset B$. Then U is open in X by (i), and thus $x \in \mathrm{int}_\xi B$. The converse follows by Lemma 7.12 (x).
To show (iii), notice that $\mathrm{bound}_\alpha B \;\subset\; \mathrm{bound}_\xi(B) \cap A$ by Lemma 7.12 (xi). Conversely, let $x \in \mathrm{bound}_\xi(B) \cap A$ and $U \in \mathcal{N}_\alpha^{\mathrm{open}}\{x\}$. It follows by (i) that $U \in \mathcal{N}_\xi^{\mathrm{open}}\{x\}$, and hence $U \cap B \neq \emptyset$ and

$$\emptyset \;\neq\; U \cap (X \backslash B) \;=\; U \cap (A \backslash B)$$

Thus $x \in \mathrm{bound}_\alpha B$. □

Lemma and Definition 7.21
Given a pseudo-metric space $\xi = (X, d)$ and a subset $A \subset X$, the restriction $d\,|(A \times A)$ is a pseudo-metric on A. It is called the **relative pseudo-metric** and denoted by $d\,|A$. The pseudo-metric space $(A, d\,|A)$ is called **pseudo-metric subspace of** ξ, or short **subspace of** ξ, and denoted by $\xi\,|A$.

Proof. This follows by Definition 5.114. □

Lemma 7.22
Given a metric space $\xi = (X, d)$ and a subset $A \subset X$, the subspace $\xi\,|\,A$ is a metric space.

Proof. This follows by Definition 5.114. □

The following Lemma states that the generation of a topology from a pseudo-metric commutes with the formation of a subspace.

Lemma 7.23
Given a pseudo-metric space $\xi = (X, d)$ and a subset $A \subset X$, we have $\tau(d)\,|A = \tau(d\,|A)$.

Proof. We define for every $r \in\ ]0, \infty[$ and $x \in X$:

$$B(x, r) = \{y \in X\ :\ d(x, y) < r\}$$

Moreover, let

$$\mathcal{B} = \{B(x, r) \cap A\ :\ r \in\ ]0, \infty[\,,\ x \in X\}$$

Then $\mathcal{B}$ is a base for for $\tau(d)\,|A$ by Lemma 7.12 (ii). Further let $d_A = d\,|A$ and

$$\mathcal{B}_A = \{B(x, r) \cap A\ :\ r \in\ ]0, \infty[\,,\ x \in A\}$$

Then $\mathcal{B}_A$ is a base for $\tau(d_A)$. We have $\mathcal{B}_A \subset \mathcal{B}$, and thus $\tau(d_A) \subset \tau(d)\,|A$. Conversely, let $r \in \,]0,\infty[\,$, $x \in X$, and $y \in B(x,r) \cap A$. Since $B(x,r)$ is $\tau(d)$-open, there is $s \in \,]0,\infty[$ such that $B(y,s) \subset B(x,r)$. Since $B(y,s) \cap A \in \mathcal{B}_A$, it follows that $\mathcal{B} \subset_\Phi \mathcal{B}_A$, and thus $\tau(d)\,|A \subset \tau(d_A)$. □

Definition 7.24

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{A} \subset \mathcal{P}(X)$. $\mathcal{A}$ is called **locally finite** if for every $x \in X$ there is a neighborhood $U \in \mathcal{N}\{x\}$ and $\mathcal{B} \sqsubset \mathcal{A}$ such that $U \cap A = \emptyset$ for every $A \in \mathcal{A} \backslash \mathcal{B}$. $\mathcal{A}$ is called **locally discrete** if for every $x \in X$ there is a neighborhood $U \in \mathcal{N}\{x\}$ and $\mathcal{B} \sqsubset \mathcal{A}$ such that $U \cap A = \emptyset$ for every $A \in \mathcal{A} \backslash \mathcal{B}$ and either $\mathcal{B} \sim 0$ or $\mathcal{B} \sim 1$. ■

Lemma 7.25

Let $(X, \mathcal{T})$ be a topological space, I an index set, and $A_i \subset X$ $(i \in I)$. If $\{A_i : i \in I\}$ is locally finite, then we have $\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$.

Proof. Assume the stated condition. Let $x \in \overline{\bigcup_{i \in I} A_i}$. There exist $U \in \mathcal{N}\{x\}$ and $J \sqsubset I$ such that $U \cap A_i = \emptyset$ for every $i \in I \backslash J$. Hence $V \cap \bigcup_{i \in J} A_i \neq \emptyset$ for every $V \in \mathcal{N}\{x\}$ by Theorem 6.71 (ii). It follows that

$$x \in \overline{\bigcup_{i \in J} A_i} = \bigcup_{i \in J} \overline{A}_i \subset \bigcup_{i \in I} \overline{A}_i$$

by Lemma 6.78 (vi). The converse follows by Lemma 6.78 (vii). □

Theorem 7.26

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}_Y)$ be topological spaces, I an index set, closed sets $A_i \subset X$ $(i \in I)$ such that $\{A_i : i \in I\}$ is locally finite and $\bigcup_{i \in I} A_i = X$, and $f : X \longrightarrow Y$ a function. f is continuous iff $f\,|A_i$ is continuous for every $i \in I$.

Proof. If f is continuous, then $f\,|A_i$ is continuous for every $i \in I$ by Lemma 7.18.

Conversely, assume that $f\,|A_i$ is continuous for every $i \in I$. For every $i \in I$ we define $\mathcal{T}_i = \mathcal{T}\,|A_i$.
Let $B \subset X$. Note that

$$\mathrm{cl}_{\mathcal{T}} B \;=\; \mathrm{cl}_{\mathcal{T}}\left(\bigcup_{i\in I}(B\cap A_i)\right) \;=\; \bigcup_{i\in I}\mathrm{cl}_{\mathcal{T}}(B\cap A_i)$$

by Lemma 7.25. Moreover, for every $i \in I$, we have $\mathrm{cl}_{\mathcal{T}(i)}(B\cap A_i) = \mathrm{cl}_{\mathcal{T}}(B\cap A_i)$ by Lemma 7.19 (ii). It follows that

$$\begin{aligned} f\,[\,\mathrm{cl}_{\mathcal{T}} B\,] \;&=\; \bigcup_{i\in I} f\,[\,\mathrm{cl}_{\mathcal{T}}(B\cap A_i)] \\ &=\; \bigcup_{i\in I}(f\,|A_i)\left[\,\mathrm{cl}_{\mathcal{T}(i)}(B\cap A_i)\right] \\ &=\; \bigcup_{i\in I}\mathrm{cl}\left((f\,|A_i)\,[B\cap A_i]\,\right) \\ &\subset\; \mathrm{cl}\left(\bigcup_{i\in I} f\,[B\cap A_i]\right) \;=\; \mathrm{cl}\,(f\,[B]) \end{aligned}$$

where the third equation is a consequence of Theorem 6.88 (ii) and the fourth line follows by Lemma 6.78 (vii). The continuity of f now follows by the same Theorem. □

Remark 7.27
Let $<$ be the standard ordering on $\mathbb{R}$, $\mathcal{T}$ the standard topology on $\mathbb{R}$, $a, b \in \mathbb{R}$ with $a < b$, $A = [a, b]$, $<_A$ the restriction of $<$ to A, $\mathcal{T}_B$ the interval topology of $(A, <_A)$, and $\mathcal{T}_A = \mathcal{T}|A$. We denote by subscript A intervals with respect to the ordering $<_A$. All other intervals refer to the ordering $<$. The system

$$\mathcal{S} = \big\{]-\infty, x[\ ,\ ]x, \infty[\ : x \in \mathbb{R} \big\} \cup \{\emptyset\}$$

is a subbase for $\mathcal{T}$. Further the system

$$\mathcal{R} = \big\{]-\infty, x[_A\ ,\ ]x, \infty[_A\ : x \in A \big\} \cup \{A\}$$

is a subbase for $\mathcal{T}_B$. We have

$$\begin{aligned} \mathcal{R} &= \big\{ [a, x[\ : x \in A\backslash\{a\} \big\} \cup \big\{]x, b]\ : x \in A\backslash\{b\} \big\} \cup \big\{A, \emptyset\big\} \\ &= \big\{ S \cap A : S \in \mathcal{S} \big\} \end{aligned}$$

It follows by Lemma 7.12 (iii) that $\mathcal{T}_A = \mathcal{T}_B$. ■

Example 7.28
Let $\xi = (\mathbb{R}, \mathcal{T})$ where $\mathcal{T}$ is the standard topology, $a, b \in \mathbb{R}$ with $a < b$, $A = \,]a, b]$, and $\alpha = (A, \mathcal{T}_A) = \xi\,|A$ the topological subspace. We define

$$\begin{aligned}
\mathcal{S}_+ &= \big\{\,]x, b] \,:\, x \in A\backslash\{b\} \,\big\} \\
\mathcal{R}_+ &= \big\{\,]x, b] \,:\, x \in \mathbb{D} \cap (A\backslash\{b\}) \,\big\} \cup \{A\} \\
\mathcal{B} &= \big\{\,]x, y[\,:\, x, y \in \mathbb{R},\, x < y \big\} \cup \{\emptyset\} \\
\mathcal{B}_A &= \big\{\,]x, y[\,:\, x, y \in A,\, x < y \big\} \cup \{\emptyset\} \\
\mathcal{A} &= \big\{\,]x, y[\,:\, x, y \in \mathbb{D},\, x < y \big\} \cup \{\emptyset\} \\
\mathcal{A}_A &= \big\{\,]x, y[\,:\, x, y \in \mathbb{D} \cap A,\, x < y \big\} \cup \{\emptyset\}
\end{aligned}$$

Each of the systems $\mathcal{B}$ and $\mathcal{A}$ is a base for $\mathcal{T}$ by Remark 5.102. Each of the systems $\mathcal{B}_A \cup \mathcal{S}_+$ and $\mathcal{A}_A \cup \mathcal{R}_+$ is a base for $\mathcal{T}_A$.
Further let $c \in \mathbb{R}$ with $a < c < b$. Then the following statements hold:

$$\begin{aligned}
&]c, b] \notin \mathcal{N}_\xi\{b\}\,, &&]c, b] \in \mathcal{N}_\alpha\{b\}\,, \\
&\mathrm{cl}_\xi\,]a, c[\;=\; [a, c]\,, && \mathrm{cl}_\alpha\,]a, c[\;=\;]a, c]\,, \\
&\mathrm{cl}_\xi\,]c, b[\;=\; [c, b]\,, && \mathrm{cl}_\alpha\,]c, b[\;=\; [c, b]\,, \\
&\mathrm{int}_\xi\, [c, b] \;=\;]c, b[\,, && \mathrm{int}_\alpha\, [c, b] \;=\;]c, b]\,, \\
&\mathrm{bound}_\xi\,]a, c[\;=\; \{a, c\}\,, && \mathrm{bound}_\alpha\,]a, c[\;=\; \{c\}
\end{aligned}$$

■

Example 7.29
Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$. The set $\mathbb{D}$ is neither $\mathcal{T}$-open nor $\mathcal{T}$-closed by Lemma 4.46. Further let $\mathcal{T}_D$ be the relative topology on $\mathbb{D}$, and $a, b \in \mathbb{R}$ with $a < b$. The set $[a, b] \cap \mathbb{D}$ is $\mathcal{T}_D$-closed. If $a, b \notin \mathbb{D}$, then this set is also $\mathcal{T}_D$-open. ■

7.3 Product topology

In this Section we consider another important special case of inverse image topologies: product topologies.

Definition 7.30
Let $\xi_i = (X_i, \mathcal{T}_i)$ $(i \in I)$ be topological spaces where I is an index set, $X = \bigtimes_{i \in I} X_i$, and $p_i : X \longrightarrow X_i$ the projections. The topology $\mathcal{T} = \tau(\{(p_i, \mathcal{T}_i) : i \in I\})$ is called **product topology** and denoted by $\prod_{i \in I} \mathcal{T}_i$. The topological space $\xi = (X, \mathcal{T})$ is called **product topological space**, or short **product space**, and denoted by $\prod_{i \in I} \xi_i$. If $I = \sigma(n) \setminus m$ for some $m, n \in \mathbb{N}$ with $m < n$, then we also write $\prod_{k=m}^{n} \mathcal{T}_k$ for $\mathcal{T}$, and $\prod_{k=m}^{n} \xi_k$ for ξ. If $I = \mathbb{N} \setminus m$ for some $m \in \mathbb{N}$, then we also write $\prod_{k=m}^{\infty} \mathcal{T}_k$ for $\mathcal{T}$, and $\prod_{k=m}^{\infty} \xi_k$ for ξ. ■

While the symbol $\bigtimes$ always denotes the Cartesian product, the symbol $\prod$ has several meanings, e.g. it denotes the product topology and the product space as well as product nets. More meanings of $\prod$ are introduced below. In each occurrence we ensure that the correct interpretation is evident from the context. We also remind the reader that we often use the notation x_i for $p_i(x)$, where $i \in I$ and $x \in X$, as introduced in the general case of generated topologies in Lemma and Definition 7.1.

By definition the projections p_i are continuous maps. Furthermore the following result holds.

Lemma 7.31

With definitions as in Definition 7.30 the projections $p_i : X \longrightarrow X_i$ are open maps.

Proof. Let $i \in I$, $U \in \mathcal{T}$, and $U_i = p_i\,[U]$. To show that U_i is open let $r \in U_i$. We may choose $x \in U$ such that $p_i(x) = r$. Further let $K \sqsubset I$ and $V_k \in \mathcal{T}_k$ $(k \in K)$ such that $x \in \bigcap_{k \in K} p_k^{-1}\,[V_k] \subset U$. If $i \notin K$, then we have $U_i = X_i$. If $i \in K$, then it follows that $r \in V_i \subset U_i$. □

Lemma 7.32

Let I be an index set. For every $i \in I$ let (X_i, R_i) be a pre-ordered space such that R_i has full field. Further let $\mathcal{T}_i$ $(i \in I)$ be the respective interval topologies and $X = \bigtimes_{i \in I} X_i$. Then, for every $i \in I$, $S_i = p_i^{-1}\,[R_i]$ is a pre-ordering on X that has full field. Let $\mathcal{S} = \{S_i : i \in I\}$, and $\mathcal{T}$ be the $\mathcal{S}$-interval topology. Then we have $\mathcal{T} = \prod_{i \in I} \mathcal{T}_i$.

Proof. For every $i \in I$, S_i is a pre-ordering by Example 2.82 and clearly has full field. We have

$$\begin{aligned} & \Big\{ \,]{-\infty}, x[_{S(i)}\,,\ ]x, \infty[_{S(i)} \,:\, x \in X,\, i \in I \Big\} \cup \{\emptyset\} \\ = \;& \Big\{ p_i^{-1}\Big[\,]{-\infty}, x_i[_{R(i)} \Big],\ p_i^{-1}\Big[\,]x_i, \infty[_{R(i)} \Big] \,:\, x \in X,\, i \in I \Big\} \cup \{\emptyset\} \\ = \;& \Big\{ p_i^{-1}\Big[\,]{-\infty}, r[_{R(i)} \Big],\ p_i^{-1}\Big[\,]r, \infty[_{R(i)} \Big] \,:\, r \in X_i\,,\, i \in I \Big\} \cup \{\emptyset\} \end{aligned}$$

The first expression is a subbase for $\mathcal{T}$, and the last is a subbase for the product topology. □

Remark 7.33
Let $\mathcal{T}$, $\mathcal{T}_+$, $\mathcal{T}^n$, and $\mathcal{T}_+^n$ be the standard topologies on $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$, and $\mathbb{R}_+^n$, respectively, where $n \in \mathbb{N}$, $n > 0$. We have $\mathcal{T}^n = \prod_{k=1}^n \mathcal{T}$ and $\mathcal{T}_+^n = \prod_{k=1}^n \mathcal{T}_+$.
■

Next we demonstrate that an iterated product of topological spaces is essentially a product of those topological spaces.

Lemma and Definition 7.34
Let $\xi_j = (X_j, \mathcal{T}_j)$ $(j \in J_i,\, i \in I)$ be topological spaces where I is an index set and J_i $(i \in I)$ are disjoint index sets. Further let $K = \bigcup \{J_i : i \in I\}$. Then $\prod_{i \in I} \left(\prod_{j \in J_i} \xi_j \right)$ and $\prod_{j \in K} \xi_j$ are homeomorphic.

Proof. For every $i \in I$ let $(Y_i, \mathcal{T}_i) = \prod_{j \in J_i} (X_j, \mathcal{T}_j)$. Further let $(Y, \mathcal{T}_Y) = \prod_{i \in I} (Y_i, \mathcal{T}_i)$ and $(X, \mathcal{T}_X) = \prod_{j \in K} (X_j, \mathcal{T}_j)$. We define the map

$$f \,:\, X \longrightarrow Y\,,$$

$$\Big(\big(f(h)\big)(i)\Big)(j) = h(j) \quad \text{for every } i \in I \text{ and } j \in J_i$$

Then f is a bijection by Remark 2.71. For every $i \in I$ the topology $\mathcal{T}_i$ is generated by $\{(p_{ij}, \mathcal{T}_j) : j \in J_i\}$ where $p_{ij} : Y_i \longrightarrow X_j$ $(j \in J_i)$ are the projections. The topology $\mathcal{T}_Y$ is generated by $\{(p_i, \mathcal{T}_i) : i \in I\}$ where $p_i : Y \longrightarrow Y_i$ $(i \in I)$ are the projections. Furthermore $\mathcal{T}_X$ is generated by $\{(q_j, \mathcal{T}_j) : j \in K\}$ where $q_j : X \longrightarrow X_j$ $(j \in K)$ are the projections. Hence

$$\mathcal{S}_X = \left\{ q_j^{-1}[U] \,:\, j \in K,\, U \in \mathcal{T}_j \right\}$$

is a subbase for $\mathcal{T}_X$. Since $q_j \circ f^{-1} = p_{ij} \circ p_i$ for every $i \in I$, $j \in J_i$, the topology $\mathcal{T}_Y$ is generated by $\left\{ \left(q_j \circ f^{-1}, \mathcal{T}_j \right) \,:\, j \in K \right\}$ by Lemma 7.7. This means that $f\, [\![\mathcal{S}_X]\!]$ is a subbase for $\mathcal{T}_Y$. Thus f is a homeomorphism. □

The following result is a direct consequence of the more general Theorem in Section 7.1, however, it is particularly important in the case of product spaces.

Corollary 7.35

With definitions as in Definition 7.30 let (x_n) be a net in X, $\mathcal{F}$ a filter on X, and $x \in X$. The following statements hold:

(i) $x_n \to x \iff \forall i \in I \quad p_i(x_n) \to p_i(x)$

(ii) $\mathcal{F} \to x \iff \forall i \in I \quad p_i [\![\mathcal{F}]\!] \to p_i(x)$

Proof. This follows by Theorem 7.9. □

Corollary 7.36

Let I be an index set. For every $i \in I$ let $(X_i, \mathcal{T}_i)$ be a topological space, $A_i \subset X_i$, and (x_n^i) a net in A_i. Further let $(X, \mathcal{T}) = \prod_{i \in I} (X_i, \mathcal{T}_i)$ and $p_i : X \longrightarrow X_i$ $(i \in I)$ be the projections. Moreover, let $(x_r) = \prod_{i \in I} (x_n^i)$ and $x \in X$. Then we have

$$x_r \to x \iff \forall i \in I \quad x_n^i \to p_i(x)$$

Proof. This is a consequence of Corollary 7.35 and Lemma 6.30. □

Theorem 7.37

Let I be an index set. For every $i \in I$ let $(X_i, \mathcal{T}_i)$ be a topological space and $A_i \subset X_i$. Moreover let $(X, \mathcal{T}) = \prod_{i \in I} (X_i, \mathcal{T}_i)$ and $p_i : X \longrightarrow X_i$ $(i \in I)$ be the projections. Further we define $A = \bigtimes_{i \in I} A_i$. The following statements hold:

(i) $\overline{A} = \bigtimes_{i \in I} \overline{A_i}$

(ii) A is closed iff A_i is closed for every $i \in I$.

(iii) $\emptyset \neq A \in \mathcal{T} \iff$
$$\big(\forall i \in I \quad \emptyset \neq A_i \in \mathcal{T}_i\big) \ \wedge\ \big(\exists K \sqsubset I \quad \forall j \in I \setminus K \quad A_j = X_j\big)$$

(iv) $A^\circ \subset \bigtimes_{i \in I} A_i^\circ$

(v) If $A^\circ \neq \emptyset$, then we have:
$$A^\circ = \bigtimes_{i \in I} A_i^\circ \iff \exists K \sqsubset I \quad \forall i \in I \setminus K \quad A_i = X_i$$

(vi) $\mathcal{T} | A = \prod_{i \in I} (\mathcal{T}_i | A_i)$

Proof. Notice that (i) follows by Theorem 6.71 (iii) and Corollaries 7.35 and 7.36. (ii) is a consequence of (i).

To show (iii), assume that $\emptyset \neq A \in \mathcal{T}$. For every $i \in I$, we clearly have $A_i \neq \emptyset$, and $A_i = p_i[A] \in \mathcal{T}_i$ since p_i is open by Lemma 7.31. Let $x \in A$. We may choose $K \sqsubset I$ and, for each $i \in K$, $U_i \in \mathcal{T}_i$ such that $x \in U \subset A$ where $U = \bigcap_{i \in K} p_i^{-1}[U_i]$. Then we obtain $X_i = p_i[U] \subset A_i$ for every $i \in I \setminus K$. The reverse implication holds by definition of the product topology.

To see (iv), let $x \in A^\circ$. There is $U \in \mathcal{T}$ such that $x \in U \subset A$. For every $i \in I$, we have $p_i(x) \in p_i[U] \subset A_i$ and $p_i[U] \in \mathcal{T}_i$ since p_i is open. Hence $p_i(x) \in A_i^\circ$ for every $i \in I$.

In order to prove (v), first note that we clearly have $A \supset \bigtimes_{i \in I} A_i^\circ$. Assume that $A^\circ \neq \emptyset$. It follows that $A_i^\circ \neq \emptyset$ for every $i \in I$ by (iv).

If there is $K \sqsubset I$ such that $A_i = X_i$ $(i \in I \setminus K)$, then $\bigtimes_{i \in I} A_i^\circ$ is open by (iii) and we obtain $A^\circ \supset \bigtimes_{i\in I} A_i^\circ$. It follows that $A^\circ = \bigtimes_{i\in I} A_i^\circ$. The converse follows by (iii).

To show (vi), let $i \in I$ and $U \in \mathcal{T}_i$. We have

$$p_i^{-1}[U] \cap A = (p_i \,|\, A)^{-1}[U \cap A_i]$$

Thus there is a subbase for $\mathcal{T}\,|A$ that is a subbase for $\prod_{i\in I}(\mathcal{T}_i\,|A_i)$ as well. □

Corollary 7.38

Let $\mathcal{T}$, $\mathcal{T}_+$, $\mathcal{T}^n$, and $\mathcal{T}_+^n$ be the standard topologies on $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$, and $\mathbb{R}_+^n$, respectively, where $n \in \mathbb{N}$, $n > 0$. We have $\mathcal{T}_+^n = \mathcal{T}^n \,|\, \mathbb{R}_+^n$.

Proof. This is shown by Remark 7.33, Lemma 7.15, and Theorem 7.37 (vi) as follows:

$$\mathcal{T}_+^n = \prod_{k=1}^{n} \mathcal{T}_+ = \prod_{k=1}^{n} (\mathcal{T}\,|\,\mathbb{R}_+) = \mathcal{T}^n \,|\, \mathbb{R}_+^n$$

□

Lemma 7.39

With definitions as in Definition 7.30, let $(Y, \mathcal{T}_Y)$ be a topological space and $g : Y \longrightarrow X$ be a map. Then g is continuous iff $p_i \circ g$ is continuous for every $i \in I$.

Proof. This follows by Theorem 7.8. □

Lemma 7.40

Let I be an index set and, for each $i \in I$, let $(X_i, \mathcal{T}_i)$ and $(Y_i, \mathcal{T}_i')$ be topological spaces and $f_i : X_i \longrightarrow Y_i$ a map. Further let $X = \bigtimes_{i\in I} X_i$ and $Y = \bigtimes_{i\in I} Y_i$, and the map $f : X \longrightarrow Y$ be defined by $\big(f(x)\big)_i = f_i(x_i)$ $(i \in I)$. Then f is continuous iff f_i is continuous for every $i \in I$.

Proof. This follows by Theorem 6.49 (vi) and Corollary 7.35 (i). □

We recall that the space of all functions from a set X to a set Y, written Y^X, is identical to the Cartesian product $\bigtimes_{x \in X} Y$ with equal factors Y. We often encounter subsets of functions $F \subset Y^X$. In such situations the following definition is convenient.

Definition 7.41

Let X and Y be two sets, $F \subset Y^X$, and $p_x : Y^X \longrightarrow Y$ $(x \in X)$ the projections, i.e. $p_x(f) = f(x)$ for every $x \in X$ and every $f \in Y^X$. Given $z \in X$, the restriction $q_z = p_z \,|\, F$ is called **evaluation function at** z. ■

Remark 7.42

With definitions as in Definition 7.41, we have $f(x) = q_x(f)$ for every $f \in F$ and $x \in X$. Let $j : F \hookrightarrow Y^X$. Then $q_x = p_x \circ j$ $(x \in X)$. ■

Since the set of functions F in Lemma and Definition 7.41 is a subset of a Cartesian product, the concepts of relative topology and product topology may be used to define a topology on F.

Definition 7.43

Let X be a set, $(Y, \mathcal{T}_Y)$ a topological space, $\mathcal{T}$ the product topology on Y^X, and $(F, \mathcal{T}_F)$ a subspace of $\left(Y^X, \mathcal{T}\right)$. $\mathcal{T}_F$ is called **topology of pointwise convergence**. ■

Remark 7.44

Let X be a set, $(Y, \mathcal{T}_Y)$ a topological space, $F \subset Y^X$, q_x $(x \in X)$ the evaluation functions with domain F, and $\mathcal{T}_F$ the topology of pointwise convergence on F. We have $\mathcal{T}_F = \tau\left(\{(q_x, \mathcal{T}_Y) : x \in X\}\right)$ by Lemma 7.7. Moreover, the functions q_x $(x \in X)$ are $\mathcal{T}_F$ -$\mathcal{T}_Y$ -continuous. ■

Example 7.45

$\mathbb{R}^{\mathbb{N}}$ is the set of all real-valued sequences. Let $\mathcal{T}$ be the topology of pointwise convergence on $\mathbb{R}^{\mathbb{N}}$ that corresponds to the standard topology on $\mathbb{R}$. The space $(\mathbb{R}^{\mathbb{N}}, \mathcal{T})$ is second countable by Remark 5.102 and Lemmas 3.70 and 3.71. ■

Theorem 7.46

With definitions as in Remark 7.44, let (f_n) be a net in F, $\mathcal{F}$ a filter on F, and $f \in F$. The following statements hold:

(i) $\big(f_n \to f \text{ with respect to } \mathcal{T}_F\big) \iff$
$\big(\forall x \in X \quad f_n(x) \to f(x) \text{ with respect to } \mathcal{T}_Y\big)$

(ii) $\big(\mathcal{F} \to f \text{ with respect to } \mathcal{T}_F\big) \iff$
$\big(\forall x \in X \quad q_x[\![\mathcal{F}]\!] \to f(x) \text{ with respect to } \mathcal{T}_Y\big)$

Proof. This is a consequence of Theorem 7.9 and Remark 7.44. □

7.4 Direct image topology

Lemma and Definition 7.47

Given a set X, topological spaces $(Y_i, \mathcal{T}_i)$ $(i \in I)$ where I is an index set, and functions $f_i : Y_i \longrightarrow X$ $(i \in I)$, the system $\bigcap_{i \in I} \{B \subset X : f_i^{-1}[B] \in \mathcal{T}_i\}$ is a topology on X. It is called **direct image topology** or the topology **generated by** $F = \{(\mathcal{T}_i, f_i) : i \in I\}$ and is denoted by $\tau(F)$. It is the finest topology $\mathcal{T}$ on X such that f_i is $\mathcal{T}_i$-$\mathcal{T}$-continuous for every $i \in I$.

Proof. $\tau(F)$ clearly has properties (i) to (iii) in Definition 5.9. Now let $\mathscr{A}$ be the set of all topologies $\mathcal{T}$ on X such that f_i is $\mathcal{T}_i$-$\mathcal{T}$-continuous for every $i \in I$. We clearly have $\tau(F) \in \mathscr{A}$. Moreover, for every $\mathcal{T} \in \mathscr{A}$ we have $\mathcal{T} \subset \tau(F)$. Hence $\tau(F)$ is the finest member of $\mathscr{A}$. □

Notice that our convention is such that every member of F has the topology of the domain space as its left coordinate, in contrast to Lemma and Definition 7.1 where the members of the generating system have the topology of the range space as their right coordinate.
The following is an important special case.

Corollary 7.48
Let X be a set, I an index set, and, for each $i \in I$, $\mathcal{T}_i$ a topology on X. Further let $F = \{(\mathcal{T}_i, \mathrm{id}_X) : i \in I\}$. We have $\tau(F) = \bigcap_{i \in I} \mathcal{T}_i$. $\tau(F)$ is the infimum of $\{\mathcal{T}_i : i \in I\}$ in the ordered space $(\mathscr{T}(X), \subset)$, i.e. it is the finest topology on X that is coarser than $\mathcal{T}_i$ for every $i \in I$.

Proof. Exercise. □

Direct image topologies may be characterized by a universal property, similarly to the case of inverse image topologies (cf. Theorem 7.8).

Theorem 7.49
Let $(X, \mathcal{T})$ be a topological space, I an index set, $(Y_i, \mathcal{T}_i)$ $(i \in I)$ topological spaces, $f_i : Y_i \longrightarrow X$ $(i \in I)$ functions, and $F = \{(\mathcal{T}_i, f_i) : i \in I\}$. The following statements are equivalent:

(i) $\mathcal{T} = \tau(F)$

(ii) For every topological space $(Z, \mathcal{T}_Z)$ and every function $g : X \longrightarrow Z$, g is $\mathcal{T}$-$\mathcal{T}_Z$-continuous iff $g \circ f_i$ is $\mathcal{T}_i$-$\mathcal{T}_Z$-continuous for every $i \in I$.

Proof. To see that (i) implies (ii), assume that $\tau(F) = \mathcal{T}$. Then f_i is $\mathcal{T}_i$-$\mathcal{T}$-continuous for every $i \in I$. Further let $(Z, \mathcal{T}_Z)$ be a topological space and $g : X \longrightarrow Z$ a map. If g is continuous, then $g \circ f_i$ is continuous for every $i \in I$ by Lemma 6.58. To show the converse let $U \in \mathcal{T}_Z$. If $g \circ f_i$ is continuous for every

$i \in I$, then we have $f_i^{-1}\left[g^{-1}[U]\right] \in \mathcal{T}_i$ $(i \in I)$, and therefore $g^{-1}[U] \in \mathcal{T}$. Thus g is continuous.

To show that (ii) implies (i), it is enough to show that the topology $\mathcal{T}$ is uniquely specified by property (ii). Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X such that (ii) is satisfied in both cases. Now let $Z = X$ and $g = \mathrm{id}_X$. Since g is $\mathcal{T}_m$-$\mathcal{T}_m$-continuous for $m \in \{1, 2\}$, it follows that f_i is $\mathcal{T}_i$-$\mathcal{T}_m$-continuous for $m \in \{1, 2\}$ and $i \in I$. Thus g is $\mathcal{T}_1$-$\mathcal{T}_2$-continuous and $\mathcal{T}_2$-$\mathcal{T}_1$-continuous, and hence $\mathcal{T}_1 = \mathcal{T}_2$. □

The following result is a characterization of the direct image topology in the case of a single function.

Theorem 7.50

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and $f : X \longrightarrow Y$ a map. If f is $\mathcal{T}_X$-$\mathcal{T}_Y$-continuous, surjective, and either $\mathcal{T}_X$-$\mathcal{T}_Y$-open or $\mathcal{T}_X$-$\mathcal{T}_Y$-closed, then we have $\mathcal{T}_Y = \tau(\{(\mathcal{T}_X, f)\})$.

Proof. Let $\mathcal{T} = \tau(\{(\mathcal{T}_X, f)\})$, and assume that f is $\mathcal{T}_X$-$\mathcal{T}_Y$-continuous, surjective, and either open or closed.

Since $\mathcal{T}$ is the finest topology on Y such that f is $\mathcal{T}_X$-$\mathcal{T}$-continuous, we have $\mathcal{T}_Y \subset \mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. Then we have $f^{-1}[U] \in \mathcal{T}_X$. First consider the case that f is $\mathcal{T}_X$-$\mathcal{T}_Y$-open. Then we have $U = f\left[f^{-1}[U]\right] \in \mathcal{T}_Y$. Second, if f is $\mathcal{T}_X$-$\mathcal{T}_Y$-closed we have

$$U = (U^c)^c = \left(f\left[f^{-1}[U^c]\right]\right)^c = \left(f\left[\left(f^{-1}[U]\right)^c\right]\right)^c \in \mathcal{T}_Y$$

□

7.5 Quotient topology

In this Section we analyse an important special case of the concept introduced in Section 7.4.

Definition 7.51
Let $(X, \mathcal{T})$ be a topological space, R an equivalence relation on X, and $f : X \longrightarrow X/R$, $f(x) = [x]$. The topology $\mathcal{T}_R = \tau(\{(\mathcal{T}, f)\})$ is called **quotient topology**. The space $(X/R, \mathcal{T}_R)$ is called **quotient topological space**, or short **quotient space**. ■

Lemma 7.52
Let X be a set, $(Y, \mathcal{T})$ a topological space, $f : Y \longrightarrow X$ a surjective function, and R the equivalence relation on Y defined by

$$(y, z) \in R \iff f(y) = f(z)$$

Further let $\mathcal{T}_R$ be the quotient topology on Y/R and $\mathcal{T}_X = \tau(\{(\mathcal{T}, f)\})$. Then $(Y/R, \mathcal{T}_R)$ and $(X, \mathcal{T}_X)$ are homeomorphic.

Proof. We define the map $g : Y \longrightarrow Y/R$, $g(y) = [y]$. Further let the map $h : Y/R \longrightarrow X$ be defined by $h([y]) = f(y)$ for every $y \in Y$. h is clearly well-defined. We show that h is a homeomorphism. h is clearly bijective. We have $h \circ g = f$. Therefore the continuity of f implies the continuity of h by Theorem 7.49. Furthermore, we have $g = h^{-1} \circ f$. Hence the continuity of g implies the continuity of h^{-1} by the same Theorem. □

Theorem 7.53
Let (X, d) be a pseudo-metric space and $R = \{(x, y) : d(x, y) = 0\}$. R is an equivalence relation on X. The map

$$D : (X/R) \times (X/R) \longrightarrow \mathbb{R}_+ , \quad D([x], [y]) = d(x, y)$$

is a metric on X/R. Moreover $\tau(D)$ is the quotient topology of $\tau(d)$.

Proof. R clearly is an equivalence relation on X. The function D is well-defined because of the triangle inequality for d. That D is a pseudo-metric follows by the fact that d is a pseudo-metric, and that D is a metric is then obvious. In order to show that D generates the quotient topology of $\tau(d)$, it is enough by Theorem 7.50 to show that the map $f : X \longrightarrow X/R$, $f(x) = [x]$, is $\tau(d)$-$\tau(D)$-continuous and $\tau(d)$-$\tau(D)$-open. Since f is an isometry, this follows by Lemma 6.66. □

Chapter 8

Functions and real numbers

In this Chapter we use the various concepts introduced in Chapters 5 to 7 (topologies, pseudo-metrics, continuity, etc.) in the context of the number systems defined in Chapter 4.

Definition 8.1

We adopt the convention that all notions related to topologies on $\mathbb{R}$ and $\mathbb{R}^n$, and their subsets refer to the respective standard topologies or their relative topologies if not otherwise specified. ■

In particular, according to this convention we refer to the standard topologies on $\mathbb{R}_+$ and $\mathbb{R}_+^n$ since these are the relative topologies by Lemma 7.15 and Corollary 7.38.

Lemma 8.2

The addition $+ : \mathbb{R}^2 \longrightarrow \mathbb{R}$, the absolute value $b : \mathbb{R} \longrightarrow \mathbb{R}_+$, the multiplication $\cdot : \mathbb{R}^2 \longrightarrow \mathbb{R}$, and, for every $m \in \mathbb{N}$, the exponentiation $h_m : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $h_m(\alpha) = \alpha^m$, are continuous functions.

Proof. We show the continuity of each function by means of Lemma 6.51. We use the bases for the respective standard topologies on $\mathbb{R}$ and $\mathbb{R}_+$ as given in Remarks 5.102 and 5.103.

The continuity of b is clear.

Let $z \in \mathbb{R}^2$, x and y be its left and right coordinates, i.e. $z = (x, y)$, (z_k) a sequence in $\mathbb{R}^2$ such that $z_k \to z$, and (x_k) and (y_k) the left and right coordinate sequences, i.e. $z_k = (x_k, y_k)$ for every $k \in \mathbb{N}$. It follows that $x_k \to x$ and $y_k \to y$ by Corollary 7.35.

To see that addition is continuous in z, let $u, v \in \mathbb{R}$ such that $u < x + y < v$. We define $w = \frac{1}{2} \min \{x + y - u,\ v - x - y\}$. There is $n \in \mathbb{N}$ such that

$$x - w < x_k < x + w, \qquad y - w < y_k < y + w$$

for every $k \geq n$. Hence we have $u < x_k + y_k < v$ for $k \geq n$.
To show that multiplication is continuous in z, let $u, v \in \mathbb{R}$ such that $u < xy < v$. We define $w = \frac{1}{2} \min \{v - xy,\, xy - u\}$. For $k \in \mathbb{N}$, we have

$$\begin{aligned} |x_k y_k - xy| &= |x_k y_k - x_k y + x_k y - xy| \\ &\leq |x_k||y_k - y| + |x_k - x||y| \end{aligned}$$

by Remark 4.48 and Lemma 5.122. We may choose $K \in \mathbb{R} \setminus \{0\}$ such that $|x| < K$. There is $n \in \mathbb{N}$ such that $|x_k| < K$, $|y_k - y| < wK^{-1}$, and, if $y \neq 0$, $|x_k - x| < w|y|^{-1}$ for every $k \geq n$. It follows that, for $k \geq n$, $|x_k y_k - xy| < 2w$, and thus $u < x_k y_k < v$.
Finally, we show that h_m is continuous for every $m \in \mathbb{N}$ by the Induction principle. The case $m = 0$ is clear. Assume that h_m is continuous for some $m \in \mathbb{N}$. The function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^2$, $f(\alpha) = (h_m(\alpha), \alpha)$, is continuous by Lemma 7.39 and Remark 7.33. Since the multiplication on $\mathbb{R}$ is continuous, the multiplication on $\mathbb{R}_+$, $g : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, $g(\alpha, \beta) = \alpha \cdot \beta$, is continuous by Corollary 7.38 and Lemma 7.18. Thus $h_{m+1} = g \circ f$ is continuous by Lemma 6.59. □

Proposition 8.3
Let $x, y \in \mathbb{R}$ and (x_n) be a sequence in $\mathbb{R}$ such that $x_n \to x$. The following statements hold:

(i) $(\forall n \in \mathbb{N} \quad x_n \leq y) \implies x \leq y$

(ii) $(\forall n \in \mathbb{N} \quad y \leq x_n) \implies y \leq x$

Proof. To see (i), we define $A = \,]-\infty, y]$. A is closed with respect to the standard topology on $\mathbb{R}$. Under the stated condition we have $x_n \in A$ for every $n \in \mathbb{N}$, and thus $x \in A$ by Lemma 6.73 (i).
The proof of (ii) is similar. □

Proposition 8.4
Let $a, b, c, w \in \mathbb{R}$ such that $\min\{a, c\} \le w \le \max\{a, c\}$. Then we have $\min\{a, b\} \le w \le \max\{a, b\}$ or $\min\{b, c\} \le w \le \max\{b, c\}$.

Proof. Exercise. □

Theorem 8.5 (Intermediate value)
Let $x, y \in \mathbb{R}$ with $x < y$, $A = [x, y]$, $f : A \longrightarrow \mathbb{R}$ a continuous function, $B = \operatorname{ran} f$, and $u = \min\{f(x), f(y)\}$, $v = \max\{f(x), f(y)\}$. The following statements hold:

(i) $u < v \implies [u, v] \subset B$

(ii) If f is strictly monotonic, then $[u, v] = B$.

(iii) We define the map $g : A \longrightarrow B$, $g(z) = f(z)$. If g is strictly monotonic, then it is bijective and g^{-1} is continuous. If g is strictly increasing (strictly decreasing), then g^{-1} is strictly increasing (strictly decreasing).

Proof. To see (i), assume that $u < v$. Let $w \in [u, v]$. We define two sequences (x_n) and (y_n) in A by $(x_0, y_0) = (x, y)$, and recursively for every $n \in \mathbb{N}$,

$$(x_{n+1}, y_{n+1}) = \begin{cases} (x_n, z) & \text{if} \quad s \le w \le t \\ (z, y_n) & \text{else} \end{cases}$$

where

$$z = \frac{1}{2}(x_n + y_n), \quad s = \min\{f(x_n), f(z)\}, \quad t = \max\{f(x_n), f(z)\}$$

It follows by the Induction principle that $y_n - x_n = (y_0 - x_0)/2^n$ for every $n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$, we have $x_n < y_n$, whence $x_n < (x_n + y_n)/2 < y_n$, and thus $x_n \le x_{n+1}$ and $y_{n+1} \le y_n$. Hence (x_n) is increasing and (y_n) is decreasing by the Induction principle. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$. Now regarding (x_n) and (y_n) as sequences in the whole of $\mathbb{R}$, they are increasing and

decreasing, respectively, and bounded. Thus they are convergent with respect to $\mathcal{T}$ by Lemma 6.10 and Remark 6.11, say $x_n \to x_\infty$ and $y_n \to y_\infty$ where $x_\infty, y_\infty \in X$. Therefore $x_\infty = y_\infty$ by Lemmas 8.2 and 6.12. Since A is $\mathcal{T}$-closed, we have $x_\infty \in A$. Again regarding (x_n) and (y_n) as sequences in A, it follows that $x_n \to x_\infty$ and $y_n \to x_\infty$ with respect to $\mathcal{T}\,|A$ by Lemma 7.16.
The continuity of f implies $f(x_n) \to f(x_\infty)$ and $f(y_n) \to f(x_\infty)$. Moreover we have for every $n \in \mathbb{N}$

$$\min\{f(x_n), f(y_n)\} \leq w \leq \max\{f(x_n), f(y_n)\}$$

by the Induction principle and Proposition 8.4. It follows that $w = f(x_\infty)$ by Proposition 8.3.
To see (ii), note that, if f is strictly monotonic, then clearly $B \subset [u, v]$.
To see (iii), notice that the map g is surjective by definition. Now assume that g is strictly monotonic. Then g is clearly injective. To see that g^{-1} is continuous, note that the system

$$\mathcal{S}_A = \big\{\, [x, z[\;:\; z \in\,]x, y] \,\big\} \cup \big\{\,]z, y] \;:\; z \in [x, y[\,\big\}$$

is a subbase for $\mathcal{T}\,|A$, and

$$\mathcal{S}_B = \big\{\, [u, z[\;:\; z \in\,]u, v] \,\big\} \cup \big\{\,]z, v] \;:\; z \in [u, v[\,\big\}$$

is a subbase for $\mathcal{T}\,|\,B$, cf. Remark 7.27. We have $g\,[\![\mathcal{S}_A]\!] \subset \mathcal{S}_B$, and thus g^{-1} is continuous by Theorem 6.49 (ii).

[If g is strictly increasing, we have for every $z \in \,]x,y]$:

$$\begin{aligned} g\big[\,[x,z[\,\big] \;&=\; g\big[\,[x,z] \setminus \{z\}\,\big] = g\big[\,[x,z]\,\big] \setminus \{g(z)\} \\ &=\; [g(x),g(z)] \setminus \{g(z)\} \;=\; [u,g(z)[\end{aligned}$$

where the third equation is a consequence of (ii) and Lemma 7.18.
Moreover if g is strictly increasing, we have for every $z \in [x,y[$:

$$g\big[\,]z,y]\,\big] = \;]g(z),v]$$

If g is strictly decreasing, we have

$$\begin{aligned} &\forall z \in \,]x,y] \quad g\big[\,[x,z[\,\big] = \;]g(z),v]\;, \\ &\forall z \in [x,y[\quad g\big[\,]z,y]\,\big] = [u,g(z)[\end{aligned}$$

]

The last two claims are now obvious. □

Corollary 8.6
Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a map with $f(0) = 0$. If f is continuous, strictly increasing, and unbounded, then f is bijective and f^{-1} is continuous and strictly increasing.

Proof. Assume the stated conditions. f is clearly injective. To see that it is surjective, let $y \in \mathbb{R}_+$. Since f is unbounded, there is $x \in \mathbb{R}_+$ such that $f(x) > y$. It follows that $y \in [0, f(x)] = f\big[\,[0,x]\,\big]$ by Lemma 7.18, and Theorem 8.5 (ii). Moreover f^{-1} is clearly strictly increasing.
Finally we show that f^{-1} is continuous. For every $m \in \mathbb{N}$ we define $A_m = [m, m+1]$ and the function

$$f_m : A_m \longrightarrow \mathbb{R}, \quad f_m(x) = f(x)$$

and $B_m = \operatorname{ran} f_m$. Clearly these maps are strictly increasing and continuous by Lemma 7.18. We have $B_m = [f_m(m), f_{m+1}(m+1)] \quad (m \in \mathbb{N})$ by Theorem 8.5 (ii). Further for every $m \in \mathbb{N}$ we define the map

$$g_m : A_m \longrightarrow B_m\,, \quad g_m(x) = f_m(x)$$

The functions g_m $(m \in \mathbb{N})$ are strictly increasing. By Theorem 8.5 (iii), for every $m \in \mathbb{N}$, g_m is bijective and g_m^{-1} is continuous. For every $m \in \mathbb{N}$ we define

$$t_m : B_m \longrightarrow \mathbb{R}_+\,, \quad t_m(y) = g_m^{-1}(y)$$

The functions t_m $(m \in \mathbb{N})$ are continuous by Lemma 7.18. We have $t_m = f^{-1}|\,B_m$ $(m \in \mathbb{N})$.

[This is seen as follows:

$$\begin{aligned}(y,x) \in t_m \quad &\Longleftrightarrow \quad (y,x) \in g_m^{-1} \quad \Longleftrightarrow \quad (x,y) \in g_m \quad \Longleftrightarrow \quad (x,y) \in f_m \\ &\Longleftrightarrow \quad (x,y) \in f \ \wedge \ x \in A_m \\ &\Longleftrightarrow \quad (y,x) \in f^{-1} \ \wedge \ y \in B_m \\ &\Longleftrightarrow \quad (y,x) \in f^{-1}|\,B_m\end{aligned}$$

]

Thus f^{-1} is continuous by Theorem 7.26. □

Lemma and Definition 8.7

For every $m \in \mathbb{N}$ with $m > 0$ let $h_m : \mathbb{R}_+ \longrightarrow \mathbb{R}_+\,,\ h_m(\alpha) = \alpha^m$. The map h_m is bijective. h_m^{-1} is called **m-th root function**. h_2^{-1} is called **square root function**. h_m^{-1} is continuous and strictly increasing for $m \in \mathbb{N}$, $m > 0$. We also write $\alpha^{\frac{1}{m}}$, $\alpha^{1/m}$, or $\sqrt[m]{\alpha}$ for $h_m^{-1}(\alpha)$. Moreover, we also write $\sqrt{\alpha}$ for $h_2^{-1}(\alpha)$. The value $\alpha^{1/m}$ is called **m-th root of** α. The value $\alpha^{1/2}$ is called **square root of** α.

For $\alpha, \beta \in \mathbb{R}_+$ and $m \in \mathbb{N}$ with $m > 0$, we have $(\alpha\,\beta)^{1/m} = \alpha^{1/m}\,\beta^{1/m}$.

Proof. Let $m \in \mathbb{N}$ with $m > 0$. Then h_m is continuous by Lemma 8.2, and strictly increasing and unbounded by Lemma 4.40. Furthermore, we have $h_m(0) = 0$. It follows by Corollary 8.6 that h_m is bijective, and that h_m^{-1} is strictly increasing and continuous.

To see the last claim notice that for every $\alpha, \beta \in \mathbb{R}_+$ and every $m \in \mathbb{N}$ with $m > 0$ we have

$$\left(\alpha^{1/m}\, \beta^{1/m}\right)^m = \alpha\, \beta$$

by Lemma and Definition 4.30. □

Definition 8.8

Let $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Further let the function $S : \sigma(n) \setminus \{0\} \longrightarrow \mathbb{R}$ be recursively defined by $S(1) = x_1$ and $S(k+1) = S(k) + x_{k+1}$ for $1 \leq k \leq n-1$. For every $m \in \mathbb{N}$ with $1 \leq m \leq n$, $S(m)$ is called a **finite series** and denoted by

$$\sum_{k=1}^{m} x_k$$

If $n \geq 2$, then we write $\sum_{k=l}^{m} x_k$ for $S(m) - S(l-1)$ where $l, m \in \mathbb{N}$ with $2 \leq l \leq m \leq n$. ■

Note that Definition 8.8 is based on the Local recursion theorem 3.52. The Recursion theorem for natural numbers, Theorem 3.13, does not suffice.

Lemma 8.9

Let $n \in \mathbb{N}$, $x, y \in \mathbb{R}^n$, and $z \in \mathbb{R}$. Then the following statements hold:

(i) $\displaystyle\sum_{k=1}^{n}(x_k + y_k) \;=\; \left(\sum_{k=1}^{n} x_k\right) + \left(\sum_{k=1}^{n} y_k\right)$

(ii) $\displaystyle\sum_{k=1}^{n}(z \cdot x_k) \;=\; z \cdot \left(\sum_{k=1}^{n} x_k\right)$

(iii) $\displaystyle x_k > 0 \;\; (1 \le k \le n) \quad\Longrightarrow\quad \sum_{k=1}^{n} x_k \;>\; 0$

(iv) $\displaystyle x_k \ge 0 \;\; (1 \le k \le n) \quad\Longrightarrow\quad \sum_{k=1}^{n} x_k \;\ge\; 0$

(v) $\displaystyle \left(x_k \ge 0 \;\; (1 \le k \le n) \;\;\wedge\;\; \sum_{k=1}^{n} x_k \;=\; 0\right) \quad\Longrightarrow\quad x_k = 0 \;\; (1 \le k \le n)$

Proof. Exercise. □

Lemma 8.10

Let $x, y \in \mathbb{R}^n$. The following statements hold:

(i) $\displaystyle \left(\sum_{k=1}^{n} x_k y_k\right)^2 \;\le\; \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)$ (Cauchy-Schwarz inequality)

(ii) $\displaystyle \left(\sum_{k=1}^{n} (x_k + y_k)^2\right)^{1/2} \;\le\; \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} y_k^2\right)^{1/2}$

Proof. We first prove (i). We have

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n}\sum_{l=1}^{n}(x_k y_l - x_l y_k)^2 = \sum_{k=1}^{n}\sum_{l=1}^{n}\left(x_k^2 y_l^2 + x_l^2 y_k^2 - 2x_k x_l y_k y_l\right) \\ &= 2\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{l=1}^{n} y_l^2\right) - 2\left(\sum_{k=1}^{n} x_k y_k\right)\left(\sum_{l=1}^{n} x_l y_l\right) \end{aligned}$$

In order to show (ii), notice that

$$\begin{aligned} \sum_{k=1}^{n}(x_k + y_k)^2 &= \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2 + 2\sum_{k=1}^{n} x_k y_k \\ &\leq \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2 + 2\left(\sum_{k=1}^{n} x_k^2\right)^{1/2}\left(\sum_{k=1}^{n} y_k^2\right)^{1/2} \\ &= \left(\left(\sum_{k=1}^{n} x_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} y_k^2\right)^{1/2}\right)^2 \end{aligned}$$

by the Cauchy-Schwarz inequality and the fact that the square root function is increasing. □

Lemma and Definition 8.11

For $n \in \mathbb{N}$ with $n > 0$ the map

$$d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_+ , \quad d(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^2\right)^{1/2}$$

is a metric. It is called **Euclidean metric**.

Proof. Notice that d satisfies the triangle inequality by Lemmas 5.122 and 8.10 (ii), and thus it is a pseudo-metric. Further d is a metric by Lemma 8.9 (v). □

Lemma 8.12
Let $n \in \mathbb{N}$ with $n > 0$, d the maximum metric on $\mathbb{R}^n$, and $\mathcal{T}$ the standard topology on $\mathbb{R}^n$. Then $\tau(d) = \mathcal{T}$.

Proof. We define the function

$$B : \mathbb{R}^n \times \,]0,\infty[\, \longrightarrow \mathcal{P}(X),$$

$$B(x,r) = \big\{ y \in \mathbb{R}^n \,:\, |x_k - y_k| < r \;\; (1 \leq k \leq n) \big\}$$

The system

$$\mathcal{B} = \big\{ B(x,r) \,:\, x \in \mathbb{R}^n,\; r \in \,]0,\infty[\, \big\} \cup \{\emptyset\}$$

is a base for $\tau(d)$ by definition. Moreover the system

$$\mathcal{A} = \big\{ \,]x,y[\, \,:\, x,y \in \mathbb{R}^n,\; (x,y) \in S \big\} \cup \{\emptyset\}$$

is a base for $\mathcal{T}$ where the interval refers to the ordering S on $\mathbb{R}^n$ as defined in Remark 5.107. We clearly have $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{A} \subset \Theta(\mathcal{B})$. ☐

Lemma 8.13
Let $n \in \mathbb{N}$ with $n > 0$. Let e be the Euclidean metric on $\mathbb{R}^n$ and d the maximum metric on $\mathbb{R}^n$. We have $\tau(d) = \tau(e)$.

Proof. Notice that we have $e(x,y) \leq \sqrt{n}\, d(x,y)$ and $d(x,y) \leq e(x,y)$ for every $x, y \in \mathbb{R}^n$. The claim follows by Lemma 5.125. ☐

Corollary 8.14
For every $n \in \mathbb{N}$ with $n > 0$, the topology generated by the Euclidean metric on $\mathbb{R}^n$ is the standard topology.

Proof. This is a consequence of Lemmas 8.12 and 8.13. ☐

Corollary 8.15

Let (x_n) be a sequence in $\mathbb{R}$. If (x_n) is convergent, then it has a unique limit point.

Proof. Let $x, y \in \mathbb{R}$ be two limit points of (x_n). For every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $|x_m - x| < 1/n$ and $|x_m - y| < 1/n$ by Remark 6.22. It follows that $|x-y| \leq |x-x_m|+|x_m-y| < 2/n$. Therefore we have $|x-y| = 0$ by Lemma 6.12 and Proposition 8.3 (ii), and thus $x = y$. □

Proposition 8.16

For every $x \in \mathbb{R}$ with $x > 1$ and every $m \in \mathbb{N}$ we have $x^m \geq m(x-1)+1$.

Proof. The claim is clear for $m = 0$ and every $x \in \mathbb{R}$ with $x > 1$. Now assume that it holds for some $m \in \mathbb{N}$ and every $x \in \mathbb{R}$ with $x > 1$. Then we have

$$\begin{aligned} x^{m+1} &= x^m \cdot x \geq \big(m(x-1)+1\big) \cdot x > m(x-1)+x \\ &= m(x-1)+(x-1)+1 = (m+1)(x-1)+1 \end{aligned}$$

□

Proposition 8.17

Let $x \in \mathbb{R}_+$. The sequence $(x^m : m \in \mathbb{N})$ is unbounded if $x > 1$, and converges to 0 if $0 < x < 1$.

Proof. If $x > 1$, then x^m is unbounded by Proposition 8.16 and Lemma 4.40. If $0 < x < 1$, then $x^m \to 0$ by Lemma 6.12. □

Lemma 8.18 (Finite geometric series)

Let $x \in \mathbb{R}$. We have

$$\sum_{k=0}^{m} x^k = \frac{1-x^{m+1}}{1-x}$$

Proof. The equality is clearly true for $m = 0$. Assuming that it is true for some $m \in \mathbb{N}$, we have

$$\sum_{k=0}^{m+1} x^k = \frac{1 - x^{m+1}}{1 - x} + x^{m+1} = \frac{1 - x^{m+2}}{1 - x}$$

□

Definition 8.19

Let (x_n) be a sequence in $\mathbb{R}$. We define the function $S : \mathbb{N} \longrightarrow \mathbb{R}$ recursively by $S(0) = x_0$, and $S(k+1) = S(k) + x_{k+1}$ $(k \in \mathbb{N})$. For every $m \in \mathbb{N}$, $S(m)$ is called a **finite series** and denoted by

$$\sum_{k=0}^{m} x_k$$

Moreover, we write $\displaystyle\sum_{k=m}^{n} x_k$ for $S(n) - S(m-1)$ where $m, n \in \mathbb{N}$ and $0 < m \leq n$.
■

Similarly to Definition 8.8, Definition 8.19 is based on the Local recursion theorem 3.52.

Lemma and Definition 8.20

Let (x_n) be a sequence in $\mathbb{R}$. If the sequence $\left(\sum_{k=0}^{m} x_k\right)_m$ is convergent, its limit point is unique and denoted by

$$\sum_{k=0}^{\infty} x_k$$

It is called **infinite series**.

Proof. The uniqueness follows by Corollary 8.15. □

Lemma 8.21 (Geometric series)
The sequence $\left(\sum_{k=0}^{m} x^k\right)_m$ has a limit point for $x \in [0,1[$. In this case we have

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Proof. We have

$$\lim_m \sum_{k=0}^{m} x^k = \lim_m \frac{1-x^{m+1}}{1-x} = \frac{1}{1-x}$$

by Lemmas 8.18 and 8.2, and Proposition 8.17. □

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